ENGINEERING MATHEMATICS - II

UNIT-I

MATRICES

SYLLABUS

- Characteristic equation
- Eigen values and eigen vectors of a real matrix
- Properties
- Cayley-Hamilton theorem
- Orthogonal transformation of a symmetric matrix to diagonal form
- Quadratic form
- Reduction of quadratic form to canonical form by orthogonal transformation.

APPLICATIONS

- In physics related applications, matrices are applied in the study of electrical circuits, quantum mechanics and optics.
- In the calculation of battery power outputs, resistor conversion of electrical energy into another useful energy, these matrices play a major role in calculations.
- Especially in solving the problems using Kirchhoff's laws of voltage and current, the matrices are essential.
- In computer based applications, matrices play a vital role in the projection of three dimensional image into a two dimensional screen, creating the realistic seeming motions.

APPLICATIONS

- Stochastic matrices and Eigen vector solvers are used in the page rank algorithms which are used in the ranking of web pages in Google search.
- The matrix calculus is used in the generalization of analytical notions like exponentials and derivatives to their higher dimensions.
- One of the most important usages of matrices in computer side applications are encryption of message codes.
- Matrices and their inverse matrices are used for a programmer for coding or encrypting a message.

DEFINITION

Matrix:

A system of equation arranged in a rectangular form along m-rows and n-columns bounded by the brackets

[ ] or ( )

$$A = \begin{bmatrix} a_{11} \ a_{12} \ \ a_{13} \ldots\ldots\ldots a_{1n} \\ a_{21} \ a_{22} \ \ a_{23} \ldots\ldots\ldots a_{2n} \\ . \\ . \\ . \\ a_{m1} \ \ a_{m2} \ a_{m3} \ldots\ldots\ldots a_{mn} \end{bmatrix}$$

TYPES OF MATRICES

- **Square matrix**

  In a matrix, number of rows is equal to number of columns (i.e) n=m is called as square matrix.

- **Row matrix**

  A matrix having a single row is called a row matrix. (i.e) (1xn)

- **Column matrix**

  A matrix having a single column is called a column matrix. (i.e) (mx1).

Types of Matrices

- **Diagonal matrix**

  In a square matrix, all the elements except the main diagonal are zeros is called a diagonal matrix.

- **Unit matrix (or) Identity matrix**

  A diagonal matrix of order n which has unity for all its diagonal elements. It is denoted by I.

- **Upper triangular matrix**

  A square matrix in which all the elements below the main diagonal elements are zeros.

- **Lower triangular matrix**

  A square matrix in which all the elements above the main diagonal elements are zeros.

TYPES OF MATRICES

- **Transpose of a matrix**

  A matrix got from any matrix A, by interchanging its rows and columns is called the transpose of the matrix and denoted by A^T

- **Symmetric matrix**

  A square matrix $A=(a_{ij})$ is said to be symmetric when $a_{ij}=a_{ji}$ for all i & j. (i.e) $A=A^T$

- **Skew Symmetric matrix**

  A square matrix $A=(a_{ij})$ is said to be Skew symmetric when $a_{ij}=-a_{ji}$ for all i & j. (i.e) $A=-A^T$

TYPES OF MATRICES

- **Singular matrix**

  A square matrix A is said to be singular if the determinant value of A is zero.

  (i.e) $|A| = 0$

CHARACTERISTIC EQUATION

- The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix *A*.

- The determinant $|A - \lambda I| = 0$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix *A*.

WORKING RULE TO FIND CHARACTERISTIC EQUATION

- Let A be any square matrix of order n. The characteristic equation of A is

$$|A - \lambda I| = 0$$

- For 2×2 matrix, the Characteristic Equation is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where

S_1 = sum of the main diagonal elements.

$S_2 = |A|$

- For 3×3 matrix

$$\lambda^3 - S_1 \lambda^2 + S_2\lambda - S_3 = 0$$

where

S_1 = sum of the main diagonal elements.

$S_2 =$ sum of the minors of main diagonal elements

$S_3 = |A|$

PROBLEM 1

- Find the Characteristic Equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\left|\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix}\right| = 0$$

$$(1-\lambda)(2-\lambda) - 0 = 0$$

$\lambda^2 - 3\lambda + 2 = 0$ is the required characteristic equation.

PROBLEM 2

- Find the Characteristic Equation of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - S_1\lambda + S_2 = 0 \text{ where}$$

$S_1 = 1 + 3$

$S_2 = |A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = -5$

Hence the C.E is $\lambda^2 - 4\lambda - 5 = 0$

PROBLEM 3

- Find the C.E of the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$.

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$$S_1 = 2 + 1 - 4 = -1$$

$S_2 =$ Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$$

Continued…

$S_2 = -2$

$$S_3 = |A| = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix} = 0$$

Hence the C.E is $\lambda^3 + \lambda^2 - 2\lambda = 0$

PRACTICE PROBLEMS

- Find the C.E of the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.
- Find the C.E of the matrix $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.
- Find the C.E of the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.
- Find the C.E of the matrix $\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & -1 & 6 \end{bmatrix}$.

Eigen Values

- Let $A = [a_{ij}]$ be a square matrix. The characteristic equation of A is $|A - \lambda I| = 0$.

- The roots of the characteristic equation are called Eigen values of A.

EIGEN VECTORS

Let $A = [a_{ij}]$ be a square matrix. If there exists a non zero vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ such that $AX = \lambda X$, then the vector X is called an **Eigen vector** of A corresponding to the **Eigen value** λ.

Working Rule to Find Eigenvalues and Eigenvectors

- Find the characteristic equation $|A - \lambda I| = 0$
- Solving the C.E we get characteristic roots. They are called Eigenvalues.
- To find Eigenvectors solve $(A - \lambda I)X = 0$ for the different values of λ.

PROBLEMS ON NON-SYMMETRIC MATRICES WITH NON-REPEATED EIGEN VALUES
PROBLEM 1

Find the Eigenvalues and Eigenvectors of the matrix $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$.

Solution:

The C.E of A is $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = 1 - 1 = 0$

$S_2 = |A| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -4$

Hence the C.E is $\lambda^2 - 4 = 0$.

Continued…

To solve the C.E $\lambda^2 = 4 \Rightarrow \lambda = \pm 2.$

Hence the Eigenvalues are $-2,2$

To find the Eigenvectors $(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (A)$$

Case I:

If λ=-2 then equation (A) becomes

$$\left[\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Continued…

$$3x_1 + x_2 = 0.$$

$$3x_1 + x_2 = 0.$$

i.e., $3x_1 + x_2 = 0$

$$3x_1 = -x_2$$

$\frac{x_1}{1} = \frac{x_2}{-3}$. Hence $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2:

If λ=2 then equation (A) becomes

$$\left[\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Continued…

$$-x_1 + x_2 = 0.$$

$$3x_1 - 3x_2 = 0.$$

$$\Rightarrow x_1 - x_2 = 0.$$

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}.$$

Hence $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

PROBLEM 2

Find the Eigenvalues and Eigenvectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$S_1 = 1 + 2 + 3 = 6$

$S_2 =$ Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

Continued…

$S_2 = 11$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = 6$$

Hence the C.E is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

If $\lambda = 1$, then $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

$\therefore \lambda = 1$ is a root.

By synthetic division,

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \multicolumn{1}{|r}{0} \end{array}$$

Continued….

Other roots are given by $\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-3)(\lambda-2)=0 \Rightarrow \lambda = 2,3$

Hence the Eigen values are $\lambda = 1,2,3.$

To find the Eigen vectors, solve $(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{(A)}$$

Case (i).

If $\lambda = 1$ then equation (A) becomes

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Continued….

$$x_3 = 0 \qquad \rightarrow (1)$$
$$x_1 + x_2 + x_3 = 0 \qquad \rightarrow (2)$$
$$2x_1 + 2x_2 + 2x_3 = 0 \qquad \rightarrow (3)$$

Since (2) and (3) are same, solving (1) and (2) by rule of cross multiplication

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 0 & & -1 & & 0 & & 0 \\ 1 & & 1 & & 1 & & 1 \end{array}$$

Continued….

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

Hence a corresponding Eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Case (ii): If λ = 2 then equation (A) becomes

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_3 = 0 \qquad \rightarrow (4)$$
$$x_1 + x_3 = 0 \qquad \rightarrow (5)$$
$$2x_1 + 2x_2 + 2x_3 = 0 \qquad \rightarrow (6)$$

Since (4) and (5) are same, solving (5) and (6) by rule of cross multiplication

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 0 & & 1 & & 1 & & 0 \\ 2 & & 1 & & 2 & & 2 \end{array}$$

Continued….

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2} \quad \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

Hence a corresponding Eigen vector is $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Case (ii). If $\lambda = 3$ then equation (A) becomes

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0x_2 - x_3 = 0 \qquad \rightarrow (7)$$

$$x_1 - x_2 + x_3 = 0 \qquad \rightarrow (8)$$

$$2x_1 + 2x_2 + 0x_3 = 0 \qquad \rightarrow (9)$$

solving (8) and (9) by rule of cross multiplication

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & & 1 & & 1 & & -1 \\ 2 & & 0 & & 2 & & 2 \end{array}$$

Continued….

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4} \quad \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

Hence a corresponding Eigen vector is $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

PROBLEMS ON NON-SYMMETRIC MATRICES WITH REPEATED EIGEN VALUES

PROBLEM 1

Find all the Eigenvalues & Eigenvectors of the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$S_1 = -2 + 1 + 0 = -1$

$S_2 =$ Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -21$$

$S_2 = -21$

$$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = 45$$

Continued…

Hence the C.E is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

If $\lambda = -3$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

$\therefore \lambda = -3$ is a root.

By synthetic division,

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \multicolumn{1}{|r}{0} \end{array}$$

Other roots are given by $\lambda^2 - 2\lambda - 15 = 0$

$$(\lambda\text{-}5)(\lambda + 3)\text{=}0 \Rightarrow \lambda = 2,3$$

Hence the Eigen values are $\lambda = -3, -3, 5.$

To find the Eigen vectors, solve $(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (A)$$

Continued…

Case (i)

If $\lambda = -3$ then equation (A) becomes

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0 \quad \rightarrow (1)$$

$$2x_1 + 4x_2 - 6x_3 = 0 \quad \rightarrow (2)$$

$$-x_1 - 2x_2 + 3x_3 = 0 \quad \rightarrow (3)$$

Since (1),(2) and (3) are same, $x_1 + 2x_2 - 3x_3 = 0$

Put $x_1 = 0,\ 2x_2 = 3x_3 \Rightarrow \frac{x_2}{3} = \frac{x_3}{2}$

Hence the corresponding Eigen vector is $X_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Continued…

Put $x_2 = 0,\ x_1 - 3x_2 = 0$

$$x_1 = 3x_2 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1}$$

Hence the corresponding Eigen vector is $X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Case (ii). If $\lambda = 5$ then equation (A) becomes

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \rightarrow (4)$$

$$2x_1 - 4x_2 - 6x_3 = 0 \quad \rightarrow (5)$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \rightarrow (6)$$

solving (4) and (5) by rule of cross multiplication

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & -3 & -7 & & 2 \\ -4 & & -6 & 2 & & -4 \end{array}$$

Continued…

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24} \quad \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence a corresponding Eigen vector is $X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

PROBLEM 2

Find all the Eigenvalues & Eigenvectors of the matrix

$$\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$$

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$$S_1 = \text{Sum of the main diagonal elements}$$

$$= 6 + (-13) + 4$$

$$= -3$$

Continued…

$$S_2 = \text{Sum of the minors of the main diagonal elements}$$

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$

$$= (-52+60) + (24-35) + (-78+84)$$

$$= 8 + (-11) + 6$$

$$= 14 - 11$$

$$= 3$$

$$S_3 = |A|$$

$$= \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$$

$$= 6(-52+60) + 6(56-70) + (-84+91)$$

$$= 6(8) + 6(-14) + 5(7)$$

$$= 48 - 84 + 35$$

$$= -1$$

Continued…

Therefore the Characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda - 1 = 0$$

To solve the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda - 1 = 0$$

$$If\ \lambda = 1\ then\ \lambda^3 + 3\lambda^2 + 3\lambda - 1 = 1 + 3 + 3 + 1 \neq 0$$

$$If\ \lambda = -1\ then\ \lambda^3 + 3\lambda^2 + 3\lambda - 1 = -1 + 3 - 3 + 1 = 0$$

$$\therefore \lambda = -1\ is\, a\, root.$$

By synthetic division,

$$\begin{array}{r|rrrr} & 1 & 3 & 3 & 1 \\ -1 & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \underline{\big|\, 0} \end{array}$$

Continued…

Other roots are given by

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$i.e., \lambda = -1 \, and \, \lambda = -1$$

Hence the Eigen values are **-1,-1,-1**

To find the Eigen vectors, solve $(A - \lambda I)X = 0$

$$(A - \lambda I)X = 0$$

$$\left[\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(A)$$

Continued…

When $\lambda = -1$ we get

$$\begin{bmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7x_1 - 6x_2 + 5x_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$14x_1 - 12x_2 + 10x_3 = 0 \ldots\ldots\ldots\ldots\ldots(2) \quad ---(B)$$

$$7x_1 - 6x_2 + 5x_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Here (1), (2) and (3) are same equations

$i.e., 7x_1 - 6x_2 + 5x_3 = 0$

Continued…

$$Put\ x_1 = 0\ in\ \ (B), we\ get$$

$$-6x_2 + 5x_3 = 0$$

$$i.e.,\ 6x_2 = 5x_3$$

$$\frac{x_2}{5} = \frac{x_3}{6}$$

$$Hence\ the\ corresponding\ Eigen\ vector\ is\ X_1 = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$$

Continued…

$$Put\ x_2 = 0\ in\ \ (B), we\ \ get$$

$$7x_1 + 5x_3 = 0$$

$$i.e.,\ 7x_1 = -5x_3$$

$$\frac{x_1}{-5} = \frac{x_3}{7}$$

$$Hence\ the\ correspondng\ Eigen\ vector\ is\ \ X_2 = \begin{bmatrix} -5 \\ 0 \\ 7 \end{bmatrix}$$

Continued…

$Put\ x_3 = 0\ in\ \ (B), we\ \ get$

$7x_1 - 6x_2 = 0 \quad i.e.,\ 7x_1 = 6x_2$

$$\frac{x_1}{6} = \frac{x_2}{7}$$

$$Hence\ the\ corresponding\ Eigen\ vector\ is\ \ X_3 = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$$

$The\ \ Eigen\ values\ are\ -1, -1, -1$

$$The\ Eigen vectors\ are\ \ X_1 = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}, X_2 = \begin{bmatrix} -5 \\ 0 \\ 7 \end{bmatrix} and\ X_3 = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$$

PROBLEMS ON SYMMETRIC MATRICES WITH NON REPEATED EIGEN VALUES

PROBLEM 1

Find the Eigenvalues & Eigenvectors of the matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$S_1 =$ Sum of the main diagonal elements

$= 7 + 6 + 5 = 18$

$S_2 =$ Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} = (30-4) + (35-0) + (42-4)$$

$$= 26 + 35 + 38 = 99$$

Continued…

$$S_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7(30-4) + 2\left(-10-0\right) + 0$$

$$= 162$$

$\therefore$ The characteric eqation is

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

Continued…

To solve the characteristic equation

$$If\ \lambda = 1\ then\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 \neq 0$$

$$If\ \lambda = -1\ then\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 \neq 0$$

$$If\ \lambda = 2\ then\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 \neq 0$$

$$If\ \lambda = -2\ then\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 \neq 0$$

$$If\ \lambda = 3\ then\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 \neq 0$$

$$\therefore \lambda = 3\ is\ a\ root\ of\ \lambda^3 - 18\lambda^2 + 99\lambda - 162$$

$$By\ synthetic\ division$$

$$\begin{array}{c|cccc} & 1 & -18 & 99 & -162 \\ 3 & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & 0 \end{array}$$

Continued…

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

$$i.e)\ (\lambda - 3)(\lambda^2 - 15\lambda + 54) = 0$$

$$(\lambda - 3)\ (\lambda - 9)\ (\lambda - 6) = 0$$

$$\lambda = 3,6,9$$

$$Hence\, the\, Eigenvalues\, of\, the\, matrix\, is\ 3,6,9$$

To find the Eigen vectors, solve (A – λI)X = 0

$$\left[\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Continued…

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots\ldots\ldots\ldots (A)$$

$Case(1).If\ \lambda = 3 then the equation (A) becomes$

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 0x_3 = 0 \qquad (1)$$

$$-2x_1 + 3x_2 - 2x_3 = 0 \qquad (2)$$

$$0x_1 - 2x_2 + 2x_3 = 0 \qquad (3)$$

Continued…

$$Solving (2)\, and\, (3)\, we\, get$$

$$\frac{x_1}{6-4} = \frac{-x_2}{-4-0} = \frac{x_3}{4-0}$$

$$\frac{x_1}{2} = \frac{-x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{-x_2}{2} = \frac{x_3}{2}$$

Hence a corresponding Eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (2). If $\lambda = 6$ then the equation (A) becomes

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Continued…

$$x_1 - 2x_2 + 0x_3 = 0 \qquad (4)$$

$$-2x_1 + 0x_2 - 2x_3 = 0 \qquad (5)$$

$$0x_1 - 2x_2 - x_3 = 0 \qquad (6)$$

Solving (5) and (6) we get

$$\frac{x_1}{0-4} = \frac{-x_2}{2-0} = \frac{x_3}{4-0}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the corresponding Eigenvector is

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Continued…

Case (3). If $\lambda = 9$ then the equation (A) becomes

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 0x_3 = 0 \qquad (7)$$

$$-2x_1 - 3x_2 - 2x_3 = 0 \qquad (8)$$

$$0x_1 - 2x_2 - 4x_3 = 0 \qquad (9)$$

solving (8) and (9) we get

Continued…

$$i.e., \frac{x_1}{12-4} = \frac{-x_2}{8-0} = \frac{x_3}{4-0}$$

$$i.e., \frac{x_1}{8} = \frac{x_2}{-8} = \frac{x_3}{4}$$

$$i.e., \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\text{Hence the corresponding Eigen vector is } X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

PROBLEMS ON SYMMETRIC MATRICES WITH REPEATED EIGEN VALUES
PROBLEM 1

Find the Eigenvalues & Eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$S_1 = 0 + 0 + 0 = 0$

$S_2 =$ Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -3$$

$S_2 = -3$

Continued…

$$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = 2$$

Hence the C.E is $\lambda^3 - 3\lambda - 2 = 0$

If $\lambda = -1$, then $\lambda^2 - 3\lambda - 2 = 0$.

$\therefore$ $\lambda = -1$ is a root.

By synthetic division,

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

Other roots are given by $\lambda^2 - \lambda - 2$=0

Continued…

$$(\lambda + 1)(\lambda - 2) = 0 \Rightarrow \lambda = -1,2$$

Hence the Eigen values are $\lambda = -1, -1, 2.$

To find the Eigen vectors, solve $(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (A)$$

Case (i).

If $\lambda = 2$ then equation (A) becomes

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0 \quad \rightarrow (1)$$

$$x_1 - 2x_2 + x_3 = 0 \quad \rightarrow (2)$$

$$x_1 + x_2 - 2x_3 = 0 \quad \rightarrow (3)$$

Continued…

solving (1) and (2) by rule of cross multiplication

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 1 & -2 & & 1 \\ -2 & & 1 & 1 & & -2 \end{array}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \quad \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence a corresponding Eigen vector is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Continued…

Case (ii):

If $\lambda = -1$ then equation (A) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0 \quad \rightarrow (4)$$

$$x_1 + x_2 + x_3 = 0 \rightarrow (5)$$

$$x_1 + x_2 + x_3 = 0 \rightarrow (6)$$

Put $x_1 = 0, \; x_2 = -x_3 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Hence the corresponding Eigen vector is $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Continued…

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 &X_2. Since the given matrix is symmetric.

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0 \rightarrow (7)$$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0 \rightarrow (8)$$

solving (7) and (8) by rule of cross multiplication

$$\begin{array}{cccc} \text{l} & \text{m} & \text{n} & \\ 1 \quad 1 & 1 & 1 & \\ 1 \quad -1 & 0 & 1 & \end{array}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1} \quad \Rightarrow \frac{l}{2} = \frac{m}{-1} = \frac{n}{-1}$$

Continued…

Hence the corresponding Eigen vector is

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

PROPERTIES OF EIGEN VALUES

- **PROPERTY : I**

(i) The sum of the Eigen values of a matrix is the sum of the elements of the main diagonal .
(or) The sum of the Eigen values of a matrix is equal to the trace of the matrix.

(ii) Product of the eigen values of a matrix is equal to the determinant of the matrix.

- **PROPERTY : 2**

A square matrix A and its transpose A^T have the same Eigen values.

PROPERTIES OF EIGEN VALUES

- **PROPERTY : 3**

If λ is an eigen value of a matrix A, then

$\frac{1}{\lambda}(\lambda \neq 0)$ is the eigen value of A^{-1}

- **PROPERTY : 4**

If λ is an eigen value of an orthogonal matrix A, then

$\frac{1}{\lambda}(\lambda \neq 0)$ is also its eigen value

- **PROPERTY : 5**

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$K\lambda_1, K\lambda_2, K\lambda_3$ are the eigen values of KA.

PROPERTIES OF EIGEN VALUES

- **PROPERTY : 6**

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$\lambda_1^p, \lambda_2^p, \lambda_3^p$ are the eigen values of A^p

- **PROPERTY : 7**

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$ are the eigen values of adj A.

- **PROPERTY : 8**

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$\lambda_1 - K, \lambda_2 - K, \lambda_3 - K$ are the eigen values of $(A\text{-}KI)$.

PROPERTIES OF EIGEN VALUES

- **PROPERTY : 9**

(i) If A is a triangular matrix then the eigen values are its diagonal elements.

(ii)If A is a singular matrix (i.e. $|A| = 0$) then one of the eigen values is 0.

- **PROPERTY : 10**

(i)If A is an orthogonal matrix then $AA^T = A^TA = I$.

(ii) If A is symmetric matrix then $A = A^T$

Problems on Properties of Eigen Values

1. The product of two Eigen values of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ is 16. Find the third Eigenvalue.

Solution :

Let the Eigenvalues of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$

Continued…

[Product of the Eigenvalues is equal to the determinant of the matrix]

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8$$

$$= 32$$

Continued…

$$\lambda_1 \lambda_2 \lambda_3 = 32$$

$$16\lambda_3 = 32$$

$$\lambda_3 = \frac{32}{16} = 2$$

PROBLEM 2

$$2.\ \text{Find the Eigen values } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ without}$$

using the characteristic equation idea.

Solution:

$$\text{Given : } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ clearly given matrix is an}$$

upper triangular matrix. Then by property, the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Hence the Eigenvalues are 2, 2, 2.

PROBLEM 3

3. Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3

and 6. Find the Eigenvalue A^{-1}.

Solution:

sum of the Eigenvalues

$$= \text{sum of the main diagonal elements}$$

$$= 3 + 5 + 3$$

$$= 11$$

Continued…

Let k be the third Eigenvalue

$$\therefore\ 3+6+k=11$$

$$9+k=11$$

$$k=2$$

Rule: If Eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3$.

then the Eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$.

$\therefore$ The Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

PROBLEM 4

If -1 is an eigen value of the matrix $A = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$, find the eigen values of A^4 using properties. (N/D 2010)

Solution:

The sum of eigen values is equal to sum of the main diagonal elements

$$\lambda_1 + \lambda_2 = 1 + 2$$

$$-1 + \lambda_2 = 3$$

$$\lambda_2 = 3 + 1$$

$$\lambda_2 = 4$$

The eigen values of A^4 are $(-1)^4$ and 4^4

that is 1, 256

PROBLEM 5

Given: $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$. **Find the eigen values of** A^2. **(Jan 2010)**

Solution:

The given matrix is the lower triangle matrix

The eigen values of A are -1,-3,2

The eigen values of A^2 are

that is 1,9,4 [by property]

PROBLEM 6

If $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ **then find the eigen values of A^{-1} and A^2-2I.**

Solution:

In a triangular matrix, the main diagonal values are the eigen values of the matrix.

$\therefore$ 2, 3, 4 are the eigen values of A.

$\therefore$ Hence the eigen values of A^{-1} are 1/2, 1/3, 1/4

The eigen values of A^2 are $(2)^2, (3)^2, (4)^2 = 4, 9, 16$.

The eigen values of A^2-2I are $4 - 2, 9 - 2, 16 - 2 = 2, 7, 14$

CAYLEY-HAMILTON THEOREM

Every square matrix A satisfies its own characteristic equation $|A-\lambda I| = 0$*, so that if*

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0$$

Uses:

(i) to calculate positive integral powers

(ii) to calculate the inverse of a square matrix.

PROBLEM 1

Verify the Cayley Hamilton Theorem for a matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \text{ and find its inverse}$$

Solution:

$$\text{Given } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Characteristics equation is $|A - \lambda I| = 0$

$(ie)\ \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To verify Cayley Hamilton Theorem we want to prove

$A^3 - 6A^2 + 9A - 4I = 0$

Continued…

$$\text{Now } A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = 0$$

$\therefore$ Cayley Hamilton Theorem is verified

Continued…

To find A $^{-1}$:

$$A^3 - 6A^2 + 9A - 4I = 0$$

(multiply by A^{-1}) $A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

PROBLEM 2

Use Cayley Hamilton Theorem find A^4 for a matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

The Characteristics equation is $|A - \lambda I| = 0$

$(ie)\ \lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0$

Continued…

By the Characteristics equation we have

$$A^3 - 4A^2 + 4A + I = 0$$

multiply by A

(ie) $A^4 - 4A^3 + 4A^2 + A = 0$

$$A^4 = 4A^3 - 4A^2 - A \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Continued…

$$\text{Now } A^2 = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{bmatrix}$$

$$\text{Using } A^3 \text{ and } A^2 \text{ in (1) we have} \quad A^4 = \begin{bmatrix} 22 & -19 & -5 \\ 24 & -9 & 14 \\ 19 & -12 & 3 \end{bmatrix}$$

PROBLEM 3

Using Cayley-Hamilton theorem, evaluate the matrix polynomial

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

for $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution:

The characteristics equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements = 5;

S_2 = Sum of the minors of the main diagonal elements = 7;

$S_3 = |A|$ =3

Continued…

The Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem , $A^3 - 5A^2 + 7A - 3I = 0$ -------(1)

Let P(A) = , $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

D(A) $= A^3 - 5A^2 + 7A - 3I$

$$
\begin{array}{r|l}
 & A^5 + A \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 - 5A^7 + 7A^6 - 3A^5 \\
 & (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & \qquad\qquad\qquad A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \qquad\qquad\qquad A^4 - 5A^3 + 7A^2 - 3A \\
 & \qquad\qquad\qquad (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & \qquad\qquad\qquad\qquad\qquad A^2 + A + I \\
\hline
\end{array}
$$

Continued…

Take $Q(A) = A^5 + A$, $R(A) = A^2 + A + I$.

We know that, $P(A) = Q(A)D(A)+R(A)$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^5 + A)(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= (A^5 + A)(0) + (A^2 + A + I) \quad \text{from (1)}$$

$$= A^2 + A + I$$

$$\text{But, } A^2 = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$\therefore P(A) = A^2 + A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

DIAGONALISATION
SIMILAR AND DIAGONALIZABLE

Two $n \times n$ matrices A and B are said to be ***similar*** if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$

This is called the ***similarity transformation***

A matrix is ***diagonalizable*** if it is similar to a diagonal matrix

ORTHOGONAL TRANSFROMATION OF A SYMMETRIC MATRIX TO DIAGONAL FORM

Let A be any square matrix of order n

Step 1: to find the characteristic equation

Step 2: to solve the characteristic equation

Step 3: to find the eigen vectors

Step 4: form the Model matrix M whose columns are the eigenvectors of A

Step 5: find N^T

Step 6: calculate AN

Step 7: calculate $D = N^T A N$

ORTHOGONAL TRANSFROMATION OF A SYMMETRIC MATRIX TO DIAGONAL FORM- PROBLEMS

Diagonalise the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and hence A^4.

Solution:

$$Let \quad A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Step1: To find the Characteristic equation

The Characteristic equation of A is $|A - \lambda I| = 0$

Continued…

$$i.e., \quad \lambda^3 - S_1\lambda^2 + S_2\lambda + S_3 = 0$$

where

$$S_1 = \text{Sum of the main diagonal elements} = 8+7+3=18$$

$$S_2 = \text{Sum of the minors of the main diagonal elements}$$

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16)+(24-4)+(56-36) = 5+20+20 = 45$$

$$S_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

Continued…

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20 = 0$$

$\therefore$ the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

Step 2 : To find the roots of the charecteristic eqn

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda\left(\lambda^2 - 18\lambda + 45\right) = 0$$

$$\lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\lambda = 0, \lambda = 3, \lambda = 15$$

Continued…

Hence the Eigen values of the matrix is 0, 3, 15

$Step\,3$: To find the Eigen vectors:

To find the Eigen vectors solve $(A-\lambda I)X=0$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Case$ (1). If $\lambda=0$ then

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Continued…

$$8x_1 - 6x_2 + 2x_3 = 0 \qquad (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \qquad (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \qquad (3)$$

$$Solving (1)\ and\ (2)\ we\ get$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Continued…

Hence the corresponding Eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (2). If $\lambda = 3$ then

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \qquad (4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \qquad (5)$$

$$2x_1 - 4x_2 + 0x_3 = 0 \qquad (6)$$

Continued…

$$Solving\,(5)\,and\,(6)\,we\,get$$

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the corresponding Eigenvector is $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Continued…

$$Solving (5)\ and\ (6)\ we\ get$$

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\text{Hence the corresponding Eigenvector is } X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Continued…

Case (3).

If $\lambda = 3$ then

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \qquad (7)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \qquad (8)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \qquad (9)$$

Continued…

$$solving(8)\, and\, (9)\ \ we\ \ get$$

$$i.e., \frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$i.e., \frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$i.e., \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\text{Hence the corresponding Eigenvector is } X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Continued…

$$solving(8)\,and\,(9)\ \ we\ \ get$$

$$i.e., \frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$i.e., \frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$i.e., \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\text{Hence the corresponding Eigenvector is } X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Continued…

$$\therefore \text{The set of Eigen vectors are,}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix},\ X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = 2 + 2 - 4 = 0$$

$$X_1^T X_3 = 2 - 4 + 2 = 0$$

$$X_2^T X_1 = 4 - 2 - 2 = 0$$

Hence the Eigen vectors are orthogonal to each other.

Continued…

$$\text{Step 4 : To form normalised matrix N.}$$

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Continued…

$$\text{Step 5: Find } N^T.$$

$$N^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\text{Step 6: Calculate } AN$$

$$AN = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Continued…

$$= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

Continued…

Step 7: Calculate $N^T AN$

$$N^T AN = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 2+2-4 & 10-20+10 \\ 0 & 4+1+4 & 20-10-10 \\ 0 & 4-2-2 & 20+20+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Continued…

$$i.e., \quad D = N^{T}AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

The diagonal elements are the Eigen values of A.

Step 8 : To find A^4

$$D = N^{T}AN$$

$$\Rightarrow A^4 = N D^4 N^T$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 15^4 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Continued…

$$=\frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 50625\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}0 & 162 & 101250\\ 0 & 81 & -101250\\ 0 & -162 & 50625\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}0+324+202500 & 0+162-202500 & 0-324+101250\\ 0+162-202500 & 0+81+202500 & 0-162-101250\\ 0-324+101250 & 0-162-101250 & 0+324+50625\end{bmatrix}$$

Continued…

$$= \frac{1}{9}\begin{bmatrix} 202824 & -202338 & 100926 \\ -202338 & 202581 & -101412 \\ 100926 & -101412 & 50949 \end{bmatrix}$$

$$= \begin{bmatrix} 22536 & -22482 & 11214 \\ -22482 & 22509 & -11268 \\ 11214 & -11268 & 5661 \end{bmatrix}$$

PROBLEM 2

Diagonalise the matrix $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ **by means of an orthogonal transformation.**

Solution:

The symmetric matrix A= $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The characteristic equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements =9;
S_2 = Sum of the minors of the main diagonal elements = 24;
$S_3 = |A|$ =16

Continued…

The characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$

$\lambda = 1, 4, 4$

Consider $\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (3-\lambda)x_1 + x_2 + x_3 &= 0 \\ 1x_1 + (3-\lambda)x_2 - 1x_3 &= 0 \\ 1x_1 - 1x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} ----(1)$$

Continued…

Case (1) : **λ = 1**

Substituting **λ=1** in (1) we get

$$2x_1 + x_2 + x_3 = 0$$

$$1x_1 + 2x_2 - 1x_3 = 0$$

$$1x_1 - 1x_2 + 2x_3 = 0$$

Solving by cross multiplication we get

$$x_1 = -1 , x_2 = 1, x_3 = 1.$$

The eigen vector is $$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Continued…

Case (2) : **λ = 4**

Substituting **λ=4** in (1) we get

$$-1x_1 + 1x_2 + x_3 = 0$$

$$1x_1 - 1x_2 - 1x_3 = 0$$

$$1x_1 - 1x_2 - 1x_3 = 0$$

All the above equations are reduced to an equation

$$x_1 - x_2 - x_3 = 0$$

Assume $x_1 = 0, x_2 = 1 \Rightarrow x_3 = -1$

The eigen vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Continued…

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

and X_3 should be orthogonal with X_1 and X_2

$$X_1^T X_3 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow -a + b + c = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow b - c = 0$$

Solving the above equations, we get $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Continued…

The Normalised modal matrix is

$$N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$N^T A N =$$

$$\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D(1,4,4)$$

QUADRATIC FORM & CANONICAL FORM

Quadratic form: A homogeneous polynomial of second degree in any number of variables.

Matrix form of the Q.F**:** Every quadratic form can be expressed as $\mathbf{X^T A X}$, where A is a symmetric matrix with a_{ii} = coefficient of x_i^2 and $a_{ij} = \left(\frac{1}{2} \times \text{coefficient of } x_i x_j\right) = a_{ji}$

Canonical form = sum of squares of any number of variable.

Matrix form of the C.F**:** Every canonical form can be expressed as $\mathbf{Y^T D Y}$**,** where D is a diagonal matrix.

RANK, INDEX, SIGNATURE OF THE QUADRATIC FORMS

Rank (r) of the Quadratic form :

The number of non zero terms in the resulting canonical form is called rank of the Quadratic form

Index (p) of the Quadratic form :

The number of positive terms in the resulting canonical form is called index of the Quadratic form

Signature of the Quadratic form :

The Signature (s) of the Quadratic form = 2p-r

NATURE OF QUADRATIC FORMS

S.No.	Nature	If the eigen values are known	If the eigen values are not known
1	Positive definite	All the eigen values are positive	D_1, D_2, D_3 are positive
2	Negative definite	All the eigen values are negative	D_1, D_3 are negative D_2 is positive
3	Positive semi definite	All the eigen values are positive and atleast one is zero	$D_1 \geq 0$, $D_2 \geq 0$ $D_3 \geq 0$ and atleast one is zero
4	Negative semi definite	All the eigen values are negative and atleast one is zero	$D_1 \leq 0$, $D_3 \leq 0$, $D_2 \geq 0$ and atleast one is zero
5	Indefinite	eigen values are positive and negative	All the other cases

where $D_1 = |a_{11}| = a_{11}$ $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION

Let A be any square matrix of order n

1. Write the matrix of the given QF
2. To find the characteristic equation
3. To solve the characteristic equation
4. To find the eigen vectors orthogonal to each other
5. Form the Normalized Model matrix N
6. Find N^T
7. $D = N^T A N$
8. Canonical from $[y1 \ \ y2 \ \ y3]\,[D]\,[y1 \ \ y2 \ \ y3]^T$

PROBLEM 1

Write down the quadratic form corresponding to the matrix

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

Solution:

The quadratic form of A is given by

$$X^T AX = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0x_1^{\ 2} + x_2^{\ 2} + 2x_3^{\ 2} + 10x_1x_2 + 12x_2x_3 - 2x_1x_3$$

PROBLEM 2

Write down the matrix form corresponding to the quadratic form

$$2x^2 + 8z^2 + 4xy + 10xz - 2yz$$

Solution:

The matrix of the quadratic form is given by

a_{11} = coeff of $x^2 = 2$, a_{22} = coeff of $y^2 = 0$, a_{33} = coeff of $z^2 = 8$

$$a_{12} = a_{21} = \frac{1}{2}(\text{coeff of } xy) = \frac{4}{2} = 2,\ a_{13} = a_{31} = \frac{1}{2}(\text{coeff of } xz) = \frac{10}{2} = 5$$

$$a_{23} = a_{32} = \frac{1}{2}(\text{coeff of } yz) = \frac{-2}{2} = -1$$

$$A = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

PROBLEM 3

Find the Rank, index and signature of the quadratic form whose canonical form is $x_1^2 + 2x_2^2 - 3x_3^2$

Solution:

Rank (r) = Number of Non zero terms in the C.F = 3

Index (p) = Number of Positive terms in the C.F = 2

Signature (s) = 2p – r =1

PROBLEM 4

Identify the Nature, Rank, Index and Signature of the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$.

Solution:

The matrix of the quadratic form is given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Continued…

The characteristics equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements = 0

S_2 = Sum of the minors of the main diagonal element

$$= (0-1) + (0-1) + (0-1) = -3$$

$S_3 = |A|$ =2

The characteristics equation is $\lambda^3 - 3\lambda - 2 = 0$.

$$\Rightarrow (\lambda + 1)^2(\lambda - 2) = 0.$$

$\therefore$ Eigen values are $\lambda = -1, -1, 2$.

Nature: indefinite

Rank (r) = Number of non zero eigen values = 3.

Index (p) = Number of positive eigen values = 1.

Signature (s) = 2p – r =2(1) – 3= –1.

PROBLEM-5

Reduce the quadratic form $8x_1^2+7x_2^2+3x_3^2-12x_1x_2-8x_2x3+4x_3x_1$ **into canonical form by means of an orthogonal transformation.**

Solution:

Given : $8x_1^2+7x_2^2+3x_3^2-12x_1x_2-8x_2x_3+4x_3x_1$

The matrix of the quadratic form is given by

$$A=\begin{pmatrix} co-eff\ of\ x_1^2 & \frac{1}{2}co-eff\ of\ x_1x_2 & \frac{1}{2}co-eff\ of\ x_1x_3 \\ \frac{1}{2}co-eff\ of\ x_1x_2 & co-eff\ of\ x_2^2 & \frac{1}{2}co-eff\ of\ x_2x_3 \\ \frac{1}{2}co-eff\ of\ x_1x_3 & \frac{1}{2}co-eff\ of\ x_2x_3 & co-eff\ of\ x_3^2 \end{pmatrix}$$

$$A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \end{pmatrix}$$

Characteristic equation:

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0 \text{.........}(1)$$

$$s_1 = \textit{sum of the leading diagonal}$$

$$s_1 = 8 + 7 + 3 = 18$$

$$s_2 = \text{sum of the minors of its leading diagonal}$$

$$s_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$s_2 = (21\text{-}16) + (24\text{-}4) + (56\text{-}36)$$

$$s_2 = 45$$

$$s_3 = Determinant\ of\ A$$

$$s_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$s_3 = 8(21-16) + 6(-18+8) + 2(24-14)$$

$$s_3 = 40 - 60 + 20$$

$$s_3 = 0$$

Substitute in (1)

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda\left(\lambda^2 - 18\lambda + 45\right) = 0$$

$$\lambda = 0; \lambda^2 - 18\lambda + 45 = 0$$

$$\lambda = 0; (\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0; \lambda = 3; \lambda = 15$$

Eigen vectors:

$$(A - \lambda I)X = 0$$

$(A - \lambda I)$ *is nothing but subtract* λ *from* A

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$Case\ (i)\ when\ \lambda = 0$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0........(1)$

$-6x_1 + 7x_2 - 4x_3 = 0.........(2)$

$2x_1 - 4x_2 + 3x_3 = 0.........(3)$

Solving any two distinct equation (2) and (3)

$$\frac{x_1}{21-16} = \frac{x_2}{-8+18} = \frac{x_3}{24-14}$$

$$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

$$when\,\lambda = 0 \;\; its\, eigen\, vector\; X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$Case \;\; (ii) \;\; when\,\lambda = 3$$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \ldots\ldots\ldots(1)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \ldots\ldots\ldots(2)$$

$$2x_1 - 4x_2 + 0x_3 = 0 \ldots\ldots\ldots(3)$$

Solving any two distinct equation (2) and (3)

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$when\, \lambda = 3 \;\; its\ eigenvector\ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$Case\ (iii) \;\; when\, \lambda = 15$$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \ldots\ldots\ldots (1)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \ldots\ldots\ldots (2)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \ldots\ldots\ldots (3)$$

.

Solving any two distinct equation (2) and (3

$$\frac{x_1}{96-16}=\frac{x_2}{-8-72}=\frac{x_3}{24+16}$$

$$\frac{x_1}{80}=\frac{x_2}{-80}=\frac{x_3}{40}$$

$$\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}$$

$$when\ \lambda=15\ \ its\ eigen\ vector\ X_3=\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

Normalized model matrix:

$$N = \begin{pmatrix} \frac{1}{\sqrt{1^2+2^2+2^2}} & \frac{2}{\sqrt{2^2+1^2+(-2)^2}} & \frac{2}{\sqrt{2^2+(-2)^2+(1)^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} & \frac{1}{\sqrt{2^2+1^2+(-2)^2}} & \frac{-2}{\sqrt{2^2+(-2)^2+(1)^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} & \frac{-2}{\sqrt{2^2+1^2+(-2)^2}} & \frac{1}{\sqrt{2^2+(-2)^2+(1)^2}} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{1}{\sqrt{9}} & \frac{2}{\sqrt{9}} & \frac{2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{1}{\sqrt{9}} & \frac{-2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{-2}{\sqrt{9}} & \frac{1}{\sqrt{9}} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \;\&\; N^T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

Consider the orthogonal transformation

$$D = N^T A N$$

$$D = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 10 & -10 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

Canonical form:

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} D \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0 y_1^2 + 3 y_2^2 + 15 y_3^2$$

PROBLEM-6

Reduce the Q.F $10x_1^2+2x_2^2+5x_3^2+6x_2x_3-10x_3x_1-4x_1x_2$.
to a canonical form and hence find rank, index, signature.

Solution:

The matrix of the Q.F is

$$A=\begin{bmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{bmatrix}$$

The C.E of A is $|A-\lambda I|=0$

$$\Rightarrow \lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

Where $s_1=10+2+5=17$

$$s_2=10-9+50-25+20-4=42$$

$$s_3=10+10-20=0$$

Continued…

$\therefore$ The characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

To find the Eigenvalues :

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0$$

$$\lambda\left(\lambda^2 - 17\lambda + 42\right) = 0$$

$$\lambda = 0, 3, 14$$

To find the Eigen vectors solve $(A - \lambda I)X = 0$

$$(10-\lambda)x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 + (2-\lambda)x_2 + 3x_3 = 0$$

$$-5x_1 + 3x_2 + (5-\lambda)x_3 = 0$$

Continued…

$$\text{Case } 1: \text{when } \lambda = 0$$

$$10x_1 - 2x_2 - 5x_3 = 0 \quad (1)$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \quad (2)$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \quad (3)$$

$$Solving(1)\, and\, (2)\, we\, get$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$Hence\, the\, corresponding\, Eigenvector\, is\, X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Continued…

$$\text{case } 2: \text{When } \lambda = 3 \text{ the eigenvector is}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \qquad (4)$$

$$-2x_1 - x_2 + 3x_3 = 0 \qquad (5)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \qquad (6)$$

$$Solving (4)\, and\, (5)\, we\; get$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$Hence\; the\; corresponding\; Eigenvector\; is\; X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Continued…

$$\text{case 3: When } \lambda = 14 \text{ the eigenvector is}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \quad (7)$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \quad (8)$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \quad (9)$$

Solving (7) and (8) we get

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\text{Hence a corresponding Eigenvector is } X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Continued…

To find model matrix

The normalised model matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix} \qquad N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

Continued…

$$N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix} = D$$

$$\text{Now } Y^T D Y = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 14y_3^2$$

which is the required canonical form.

Continued…

$$\text{Rank} = 2$$

$$\text{Index} = 2$$

$$\text{Signature} = 2$$

$$\text{Nature} = \text{Positive semi definite}$$


Thank You!