Derivatives:

Definition: Tangent line

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line tangent P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided that this limit exists.

(or)

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition: Derivative

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \frac{f(a+h) - f(a)}{h}$$ if this limit exists.

(or) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Note: $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$.

1. Find the equation of the tangent line to the Parabola $y = x^2 - 8x + 9$ at the Point $(3, -6)$.

Soln: Given: $y = f(x) = x^2 - 8x + 9$

$$\frac{dy}{dx} = 2x - 8$$

$$\left(\frac{dy}{dx}\right)_{at(3,-6)} = 2(3)-8 \qquad \left[\because y' = dy/dx = m\right.$$

$$= 6-8$$

$$= -2$$

$$\therefore \boxed{m=-2}$$

The equation of the tangent line at $(3,-6)$

$$\boxed{y-y_1 = m(x-x_1)} \qquad \left[\because \begin{matrix}(x_1, y_1)\\(3,-6)\end{matrix}\right.$$

$$y-(-6) = (-2)(x-3)$$

$$y+6 = -2x+6$$

$$y = -2x+\cancel{6}-\cancel{6}$$

$$\therefore \boxed{y=-2x} \Rightarrow 2x+y=0$$

---

2. Find the slope of the tangent line to the Parabola $y = 4x-x^2$ at $(1,3)$.

Soln Given: $y = 4x-x^2$

$$dy/dx = y' = f'(x) = 4-2x$$

$$\left(dy/dx\right)_{at(1,3)} = 4-2(1) = 4-2 = 2$$

$$\therefore \boxed{m=2}$$

---

3. Find the equation of the tangent line to the curve at the given point $y = 3/x$, $(3,1)$.

Soln:

$$y = f(x) = 3/x, \quad (3,1)$$

$$m = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x\to 3} \frac{3/x - 1}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{3-x}{x}}{x-3} = \lim_{x\to 3} \frac{3-x}{x(x-3)}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{x\cancel{(x-3)}}$$

$$\therefore m = \lim_{x\to 3} -1/x = -1/3$$

The equation of the tangent line at $(3,1)$ (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -1/3\,(x-3)$$

$$y - 1 = -1/3\,x + 1$$

$$3y - 3 = -x + 3$$

$$\Rightarrow 3y = -x + 6$$

$$\Rightarrow y = -x/3 + 6/3$$

$$\therefore \boxed{y = -x/3 + 2}$$

4. If $x^3 - x$, then find $f'(x)$ & $f''(x)$.

Soln:

Given: $f(x) = x^3 - x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - (x^3 - x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - (3x^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x^2 + h^2 + 2xh) - 1] - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 3h^2 + 6xh - \cancel{1} - \cancel{3x^2} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + 6xh}{h} = \lim_{h \to 0} \frac{\cancel{h}(3h + 6x)}{\cancel{h}}$$

$$\therefore f''(x) = \lim_{h \to 0} (3h + 6x) = 6x$$

5. Find the equation of the tangent line and normal line to the curve at the given point

$y = 3x^2 - x^3, \ (1, 2).$

Soln

Given: $y = 3x^2 - x^3$

$$y' = \frac{dy}{dx} = 6x - 3x^2$$

$$\left(\frac{dy}{dx}\right)_{(1,2)} = 6(1) - 3(1)^2 = 6 - 3 = 3.$$

$\therefore \ m = 3$ (slope)

Equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$y = 3x - 3 + 2$$

$$\therefore \boxed{y = 3x - 1}$$

Equation of the Normal line is

$$y - y_1 = -1/m \, (x - x_1)$$

$$y - 2 = -1/3 \, (x - 1)$$

$$3y - 6 = -x + 1$$

$$3y = -x + 1 + 6$$

$$3y = -x + 7$$

$$\boxed{y = -x/3 + 7/3}$$

6. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

Soln: Given: $y = 2x^3 + 3x^2 - 12x + 1$

Horizontal tangent occur where the derivative is zero. (ie) $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 + 6x - 12 = 0$$

$$\Rightarrow 6(x^2 + x - 2) = 0$$

$$\Rightarrow x^2 + x - 2 = 0$$

-2

$-1 \quad 2$

$$\therefore x = 1, -2$$

$\therefore$ The curve $y = 2x^3 + 3x^2 - 12x + 1$ has horizontal tangents $x = 1, -2$.

The corresponding points on the curve are $(1, -6), (-2, 21)$

H.W Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent? If so where?

Ans: $x = 0, 1, -1$

$(0, 2)$, $(1, 1)$ and $(-1, 1)$

Implicit Differentiation:-

$$\boxed{\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}}$$

1. (i) Find $\frac{dy}{dx}$ if $x^3+y^3 = 6xy$.

(ii) Find the tangent to the equation $x^3+y^3=6xy$ (this equation is known as folium of Descartes) at the point $(3,3)$.

(iii) At what point in the first quadrant is the tangent line horizontal.

Soln (i) Given: $x^3+y^3 = 6xy \longrightarrow (1)$

(ie) $x^3+y^3-6xy=0$

Let $u = x^3+y^3-6xy$

$$\frac{\partial u}{\partial x} = 3x^2-6y$$

$$\frac{\partial u}{\partial y} = 3y^2-6x$$

$$\therefore \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = -\frac{3x^2-6y}{3y^2-6x}$$

$$= -\frac{\cancel{3}(x^2-2y)}{\cancel{3}(y^2-2x)}$$

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

(ii) $\left(\frac{dy}{dx}\right)_{(3,3)} = \frac{2(3)-(3)^2}{(3)^2-2(3)}$

$= \frac{6-9}{9-6} = \frac{-3}{3} = -1.$

$\therefore \boxed{m=-1}$

$\therefore$ The equation of the tangent at the Point $(3,3)$ is

$y-y_1 = m(x-x_1)$

$y-3 = -1(x-3)$

$y-3 = -x+3$

$y = -x+3+3$

$y = -x+6$

$\therefore \boxed{x+y=6}$

(iii) The tangent line is horizontal if $y'=0$ (or) $dy/dx = 0$

$\frac{2y-x^2}{y^2-2x} = 0$

(ie) $2y-x^2=0$ and $y^2-2x \neq 0$

$\therefore 2y-x^2=0$

$\Rightarrow 2y = x^2$

$\Rightarrow y = x^2/2.$

Sub. $y = x^2/2$ in eqn ①, we get

$\therefore$ ① $\Rightarrow x^3 + y^3 = 6xy$

$\Rightarrow x^3 + \left(\frac{x^2}{2}\right)^3 = \cancel{6}^{3} x \left(\frac{x^2}{\cancel{2}}\right)$

$\Rightarrow x^3 + \frac{x^6}{8} = 3x^3$

$\Rightarrow 8x^3 + x^6 = 24x^3$

$\Rightarrow x^6 = 24x^3 - 8x^3$

$\Rightarrow x^6 = 16x^3$

$x^3 = 16$ $\quad$ $\left[\because (2^4)^{1/3} = 2^{4/3}\right.$

$x = (16)^{1/3} = 2^{4/3}$

$\therefore \boxed{x = 2^{4/3}}$

$y = \frac{(2^{4/3})^2}{2} = \frac{2^{8/3}}{2} = 2^{8/3} \cdot 2^{-1} = 2^{8/3 - 1}$

$\therefore \boxed{y = 2^{5/3}}$

The tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately $(2.5198, 3.1748)$

—————— o ——————

2. Use implicit differentiation to find an equation of the tangent line to the curve at the given point

$$y \sin 2x = x \cos 2y, \quad (\pi/2, \pi/4).$$

Soln.

Given: $y \sin 2x = x \cos 2y$

$$y(2\cos 2x) + \sin 2x \frac{dy}{dx} = x\left(-2\sin 2y \frac{dy}{dx}\right) + \cos 2y (1)$$

$$\left(\sin 2x + 2x \sin 2y\right)\frac{dy}{dx} = \cos 2y - 2y \cos 2x$$

$$\frac{dy}{dx} = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x \sin 2y}$$

$$\left(\frac{dy}{dx}\right)_{(\pi/2, \pi/4)} = \frac{\cos 2(\pi/4) - 2(\pi/4)\cos 2(\pi/2)}{\sin 2(\pi/2) + 2(\pi/2)\sin 2(\pi/4)}$$

$$= \frac{0 - 2(\pi/4)(-1)}{0 + 2(\pi/2)(1)}$$

$$= \frac{\cancel{2}(\pi/4)}{\cancel{2}(\pi/2)} = \frac{\cancel{\pi}}{\cancel{4}_2} \times \frac{\cancel{2}^1}{\cancel{\pi}}$$

$$\therefore \boxed{m = 1/2}$$

$\therefore$ Equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - \pi/4 = 1/2 (x - \pi/2)$$

$$y - \pi/4 = 1/2 x - \pi/4$$

$$y = 1/2 x - \pi/4 + \pi/4$$

$$\therefore \boxed{y = 1/2 x} //.$$

3. Find the first two derivatives for $x^4+y^4=16$.

Soln: Let $u = x^4+y^4-16$

$$\frac{\partial u}{\partial x} = 4x^3$$

$$\frac{\partial u}{\partial y} = 4y^3$$

$$\therefore \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

$$\therefore \boxed{y' = \frac{dy}{dx} = \frac{-x^3}{y^3}}$$

Again diff. w.r.to 'x' on both sides

$$\frac{d^2y}{dx^2} = -\frac{y^3(3x^2) - x^3(3y^2)\,dy/dx}{(y^3)^2}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-x^3/y^3\right)}{y^6}$$

$$= -\frac{3x^2y^3 + 3x^6/y}{y^6}$$

$$= -\frac{3x^2y^4 + 3x^6}{y^7}$$

$$= -\frac{3x^2(y^4+x^4)}{y^7}$$

$$\therefore y'' = d^2y/dx^2 = -\frac{3x^2(16)}{y^7} = -\frac{48x^2}{y^7}$$

UNIT-1
DIFFERENTIAL CALCULUS

SYLLABUS

- REPRESENTATION OF FUNCTIONS
- LIMIT OF A FUNCTION
- CONTINUITY OF A FUNCTION
- DERIVATIVES OF A FUNCTION
- MAXIMA AND MINIMA OF A FUNCTION OF SINGLE VARIABLE.

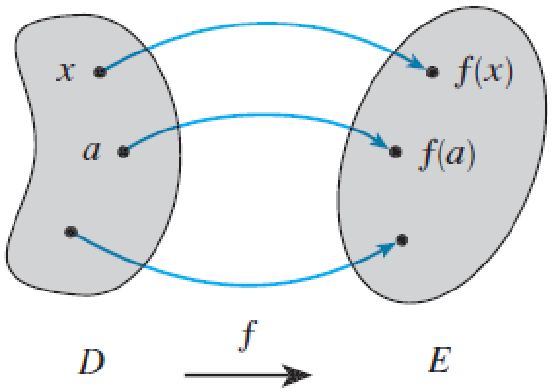
- Objective: To provide the basic tools of calculus mainly for the purpose of modelling the engineering problems mathematically and obtaining solutions. Single variable and multivariable calculus plays an important role in the understanding of science, engineering, economics and computer science, among other disciplines.

- Outcomes: Able to use both the limit definition and rules of differentiation to differentiate functions and apply differentiation to solve maxima and minima problems.

Definition

Machine diagram for a function f

Arrow diagram for f



A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

The set D is called the **domain** of the function.

The number $f(x)$ is the value of f at x and is read "f of x."

The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A symbol that represents an arbitrary number in the ***domain*** of a function f is called an **independent variable**.

A symbol that represents a number in the ***range*** of f is called a **dependent variable**.

The domain conversion

Function	Domain (x)	Range (y or $f(x)$)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	R – {0} Non zero Real numbers	R – {0}
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \sin x$	$(-\infty, \infty)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ principal domain	$[-1, 1]$
$y = \cos x$	$(-\infty, \infty)$ $[0, \pi]$ principal domain	$[-1. 1]$
$y = \tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ principal domain	$(-\infty, \infty)$
$y = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = \log_e^x$	$(0, \infty)$	$(-\infty, \infty)$

DIFFERENTIAL CALCULUS

1. Find the domain of each function.

$$\text{(a)}\ f(x)=\sqrt{x+2} \quad \text{(b)}\ g(x)=\frac{1}{x^2-x}$$

Solution:

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$.

This is equivalent to $x \geq -2$, so the domain of f(x) is the interval $[-2, \infty)$.

(b) Since $g(x)=\dfrac{1}{x^2-x}=\dfrac{1}{x(x-1)}$

When x = 0 or x = 1 g(x) is not defined. Therefore the domain of g(x) is the interval $(-\infty, 0)\cup(0, 1)\cup(1, \infty)$

Definition

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

LIMIT OF A FUNCTION

A **limit** is the idea of looking at what happens to a function as you ***approach*** particular values of x.

A limit is strictly the behavior of a function "**near**" a point.

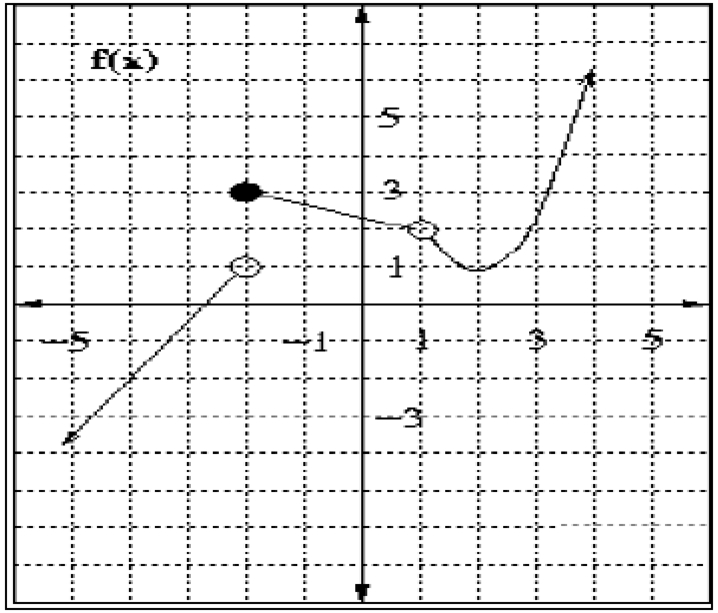
Left-hand and right-hand limits are the idea of looking at what happens to a function as you approach a particular value of x, from a *particular direction*.

The **left hand limit** is the limit of *f(x)* as x approaches the value of a from the left is written $\lim_{x \to a^-} f(x)$

The **right hand limit** is the limit of *f(x)* as x approaches the value of a from the right is written $\lim_{x \to a^+} f(x)$

EXAMPLE

Example 1 : Consider the graph of $f(x)$ in the given figure below.



The following observations can be made

when $x = -2$, we notice there is a "break" in the function.

EXAMPLE

However, if we approach $x = -2$ “from the left” (Figure a) we can see that the function values are getting closer and closer to 1.

On the other hand, if we approach $x = -2$ “from the right” (Figure b) we can see that the function values are getting closer and closer to 3.

Figure a

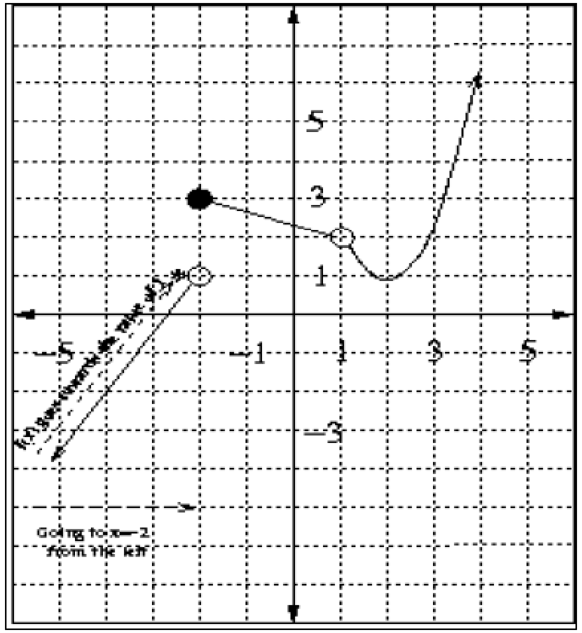
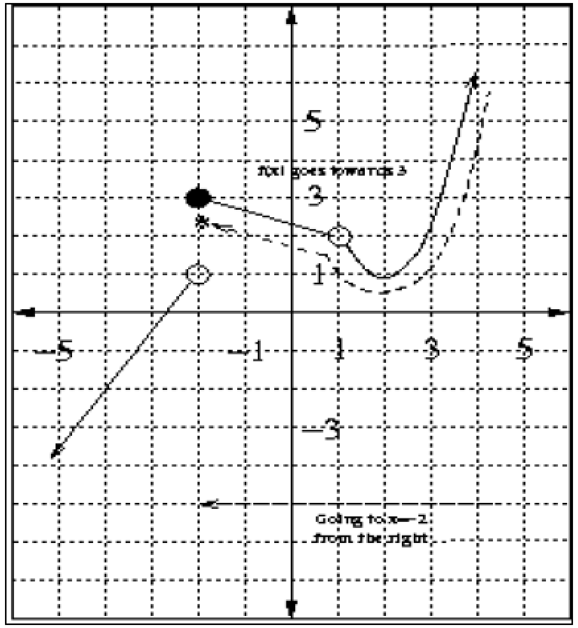


Figure b

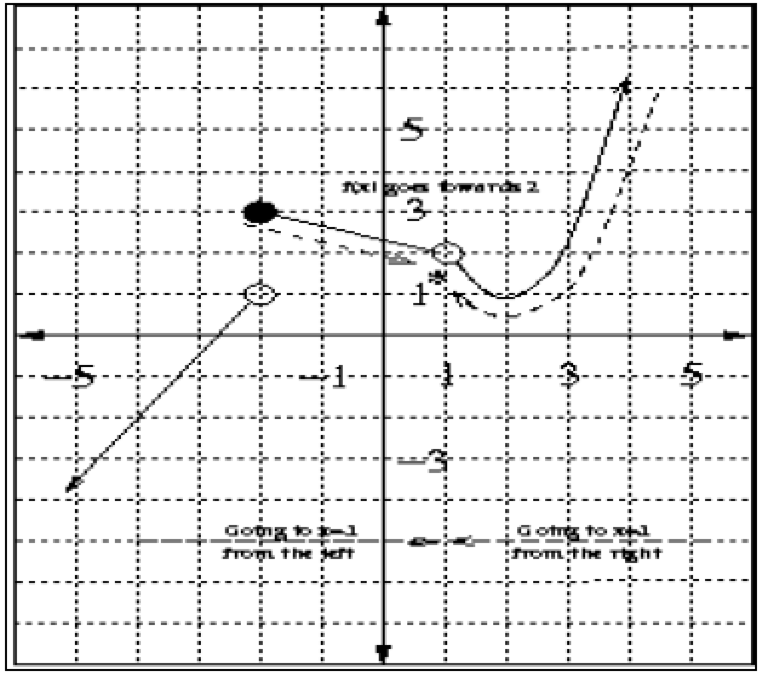


When $x = 1$, we notice there is a hole in the function.

EXAMPLE

If we approach *f(x)* from the left *or* from the right (Figure c), we can see that the function values are getting closer and closer to 2.

Figure c

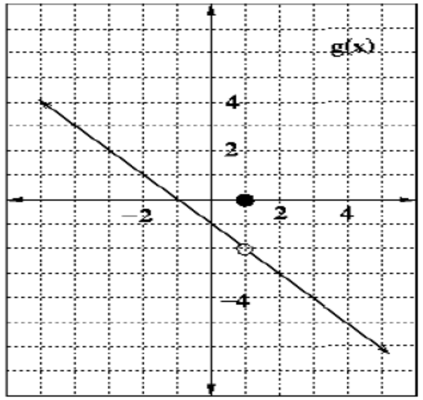


Therefore, we have $\lim_{x\to -2^-} f(x)=1$ $\lim_{x\to -2^+} f(x)=3$ $\lim_{x\to 1^-} f(x)=2$ $\lim_{x\to 1^+} f(x)=2$

EXAMPLE

2. Using the given graph of *g(x)*, find the following left- and right-hand limits.

$$\lim_{x \to 0^-} g(x), \quad \lim_{x \to 0^+} g(x), \quad \lim_{x \to 1^-} g(x), \quad \lim_{x \to 1^+} g(x).$$



Solution:

The graph of *g(x)* implies as *x* approaches *0* from the left, we can see that the function values are getting closer and closer to -*1*. (Fig a) So, $\lim_{x\to 0^-} g(x) = -1$

As *x* approaches *0* from the right, we can see that the function values are getting closer and closer to -*1*.

(Fig b)So, $\lim_{x\to 0^+} g(x) = -1$

PROBLEM

As x approaches 1 from the left, we can see that the function values are getting closer and closer to -2.

c) So, $\lim_{x \to 1^-} g(x) = -2$

As x approaches 1 from the right. You can see that the function values are getting closer and closer to -2.

(Fig d) So, $\lim_{x \to 1^+} g(x) = -2$

Fig a

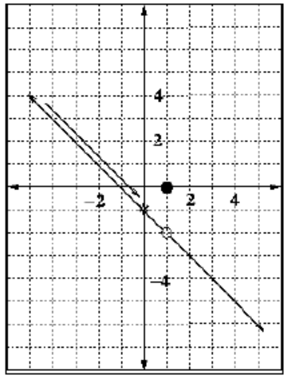


Fig b

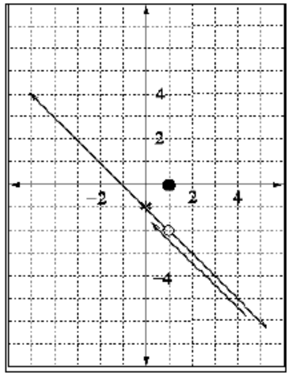


Fig c

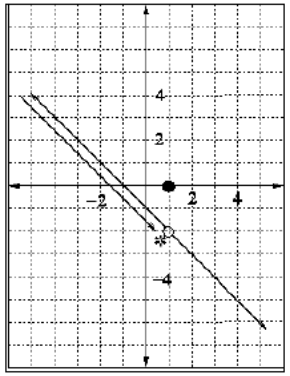
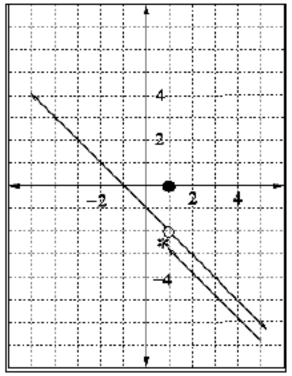


Fig d



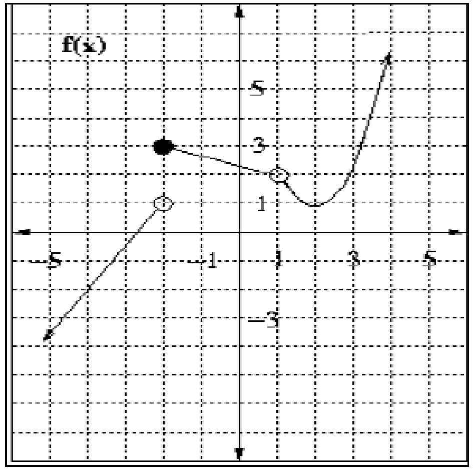
By considering both the left- and right-hand limits of a function as you approach a particular value of x, you can determine whether or not the limit of the function at that point exists.

EXAMPLE

DEFINITION : For a given function $y = f(x)$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$, finite answer, we say that **limit exists** at $x = a$ and we write that $\lim_{x \to a} f(x) = l$.

Otherwise limit does not exist.

Example 2: Using the graph of *f(x)* below, find the following limits. $\lim_{x \to 1} f(x)$, $\lim_{x \to -2} f(x)$.



Already in example 1, we have got $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 2$

Therefore, by definition, since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$, then $\lim_{x \to 1} f(x) = 2$.

PROBLEM

3. Find $\lim_{x \to 1} \frac{1}{x-1}$, if it exists

Solution: Consider $f(x) = \frac{1}{x-1}$

X	.9	0.99	0.999	0.9999	1.0	1.0001	1.001	1.01	1.1
f(x)	-10	-100	-1000	-10000	-	10000	1000	100	10

By the observation of the values $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$ and $\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$.

Left hand limit $\neq$ Right hand limit

Thus limit does not exist.

PROBLEM

4. Use conjecture method to determine the limit $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Solution: Consider f(x)= $\frac{e^x - 1}{x}$

X	-1	-0.1	-0.001	-0.0001	0.0	0.0001	0.001	0.1	1
f(x)	0.632	0.956	0.999	1.000	-	1.000	1.001	1.052	1.718

From the table it appears that as x gets closer to 0, the function f(x) gets closer to 1.

Therefore we conclude $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

PROBLEM

5. Show that $\lim\limits_{x \to 0} |x| = 0$

Solution: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Since $|x| = x$, for $x > 0$,

$$\therefore \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$, $|x| = -x$,

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} (-x) = 0$$

$$\therefore \lim_{x \to 0^+} |x| = \lim_{x \to 0^-} |x| = \lim_{x \to 0} |x| = 0$$

PROBLEM

6. Prove that $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist

Solution: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

For $x > 0$, $|x| = x$, $\therefore \lim\limits_{x \to 0^+} \frac{|x|}{x} = \lim\limits_{x \to 0^+} \frac{x}{x} = 1$

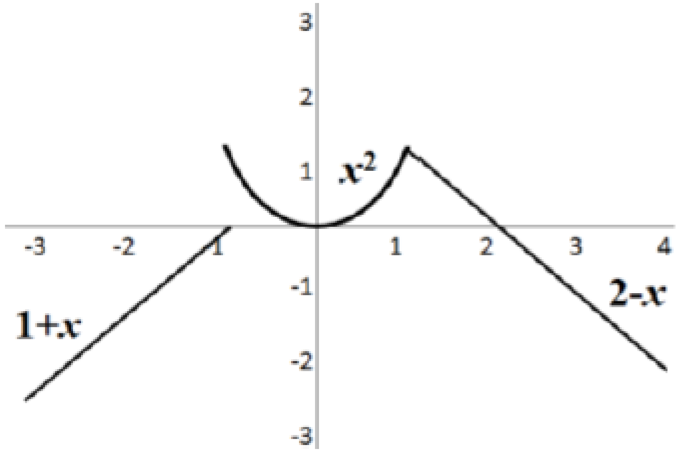
For $x < 0$, $|x| = -x$, $\therefore \lim\limits_{x \to 0^-} \frac{|x|}{x} = \lim\limits_{x \to 0^-} \frac{-x}{x} = -1$

$\therefore \lim\limits_{x \to 0^+} \frac{|x|}{x} \neq \lim\limits_{x \to 0^-} \frac{|x|}{x}$ $\therefore$ limit does not exist.

PROBLEM

7. Sketch the graph of the function $f(x)=\begin{cases} 1+x; & x<-1 \\ x^2; & -1\le x\le 1 \\ 2-x; & x\ge 1 \end{cases}$ and use it to determine the value of "a" for which $\lim\limits_{x\to a} f(x)$ exists?

Solution:



From the graph $\lim\limits_{x\to a} f(x)$ exists for all 'a' except when a = -1,

$\Rightarrow$ the left and right limits are different at a = -1

PROBLEM

8. Find $\lim_{x \to 0} \frac{1}{x^2}$

Solution:

x	-1	-0.5	-0.2	-0.1	-0.05	-0.01	-0.001	0	1	0.5	0.2	0.1	0.05	0.01	0.001
f(x)	1	4	25	100	400	10,000	1,000,000	-	1	4`	25	100	400	10,000	1,000,000

As x becomes close to 0, x^2 also becomes close to 0, and $\frac{1}{x^2}$ becomes very large. Thus the values of f(x) can be made arbitrarily large by taking x close enough to 0.

Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

PROPERTIES

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

PROBLEMS

9. Evaluate $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x^3 - 1}$, if it exists.

Solution:

$$\lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x^2 - 1)(x^2 + 1)}{(x - 1)(x^2 + x + 1)}$$

$$= \lim_{x \to 1} \frac{(x + 1)(x - 1)(x^2 + 1)}{(x - 1)(x^2 + x + 1)}$$

$$= \lim_{x \to 1} \frac{(x + 1)(x^2 + 1)}{(x^2 + x + 1)} = \frac{(1 + 1)(1 + 1)}{(1 + 1 + 1)} = \frac{4}{3}$$

PROBLEMS

10. Evaluate $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

Solution:

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \lim_{t \to 0} \frac{t^2+9-9}{t^2} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{6}$$

EXCERCISES

11. $$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}(x+2)=4$$

12. $$\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}(x^2+2x+4)=12$$

13. $$\lim_{x\to 2}\frac{x^2-5x+6}{x-2}=\lim_{x\to 2}(x-3)=-1$$

14. $$\lim_{x\to\infty}\frac{x^2-4x+3}{x^3-2}=\lim_{x\to\infty}\frac{x^2}{x^3}=\lim_{x\to\infty}\frac{1}{x}=0$$

15. $$\lim_{x\to\infty}\frac{x^2-4x+3}{x^2-2}=\lim_{x\to\infty}\frac{x^2}{x^2}=1$$

16. $$\lim_{x\to\infty}\frac{x^4-4x+3}{x^3-2}=\lim_{x\to\infty}\frac{x^4}{x^3}=\lim_{x\to\infty}\frac{x}{1}=\infty \text{ ,or Does Not Exist.}$$

SANDWICH THEOREM

The Squeeze theorem (or) The Sandwich Theorem (or) The Pinching Theorem:

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$

Show that $\mathbf{\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0}$

Solution: Let $g(x) = x^2 \sin\left(\frac{1}{x}\right)$

Since $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist

$\therefore$ we cannot use $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \,.\, \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

Since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}\left(-x^2\right) = 0$

$\therefore \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

CONTINUITY OF A FUNCTION

Definition:

A function $f(x)$ is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

A function $f(x)$ is said to be **discontinuous** at a if it is not continuous at a.

Function f has a **discontinuity** at a if any one of the following occurs.

(i) $f(a)$ is not defined in the domain of f.

(ii) $\lim_{x\to a} f(x)$ does not exist or $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.

(iii) $\lim_{x\to a} f(x) \neq f(a)$

A function f is **continuous from the right** at a number a if $\lim_{x\to a^+} f(x) = f(a)$.

A function f is **continuous from the left** at a number a if $\lim_{x\to a^-} f(x) = f(a)$.

A function f is said to be **continuous in an interval** $[a,b]$ if it is continuous at each and every point of the interval.

Note: If f is defined only on one side of end point of the interval $[a,b]$ then **continuous at the end point** mean f is continuous from the right or continuous from the left. i.e. $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

Examples of continuous function

The following functions are continuous at every point in their domains:

- ✓ constant functions
- ✓ polynomials
- ✓ rational functions
- ✓ root functions
- ✓ trigonometric functions
- ✓ inverse trigonometric functions
- ✓ exponential functions
- ✓ logarithmic functions

Results of continuous function

If $f(x)$ and $g(x)$ are two continuous functions at a and if c is any constant then the following functions are also continuous at a .

(i) $f(x)+g(x)$

(ii) $f(x)-g(x)$

(iii) $c\,f(x)$

(iv) $f(x)\,g(x)$

(v) $\dfrac{f(x)}{g(x)}, g(x) \neq 0.$

(vi) If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.

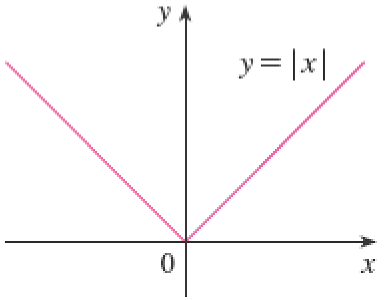
In other words, $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

(vii) If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

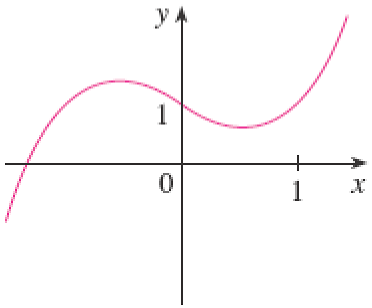
Graphical representation of continuous function

The graph of continuous function can be drawn without removing your pen from the paper.

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

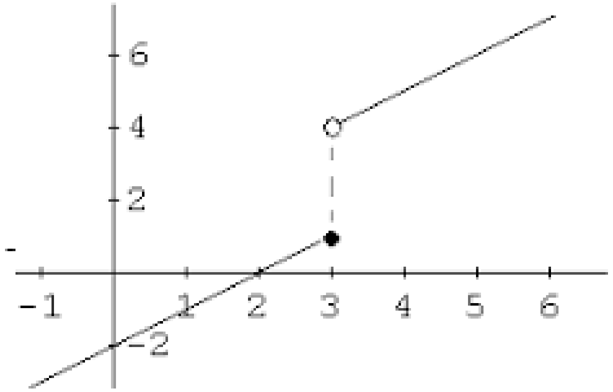


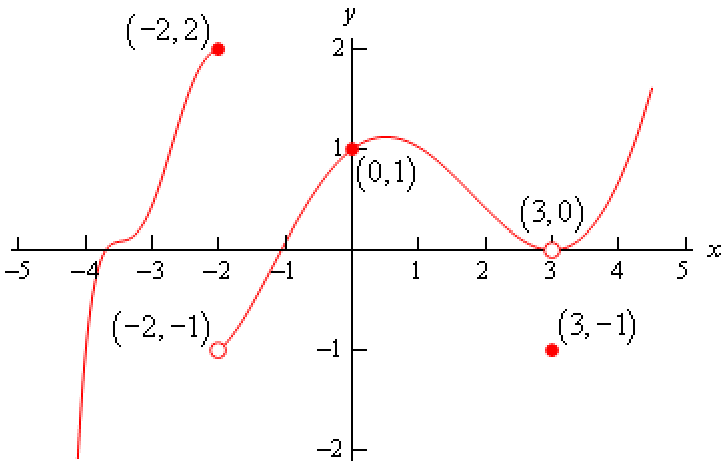
$$y = x^3 - x + 1$$



Graphical representation of discontinuous function

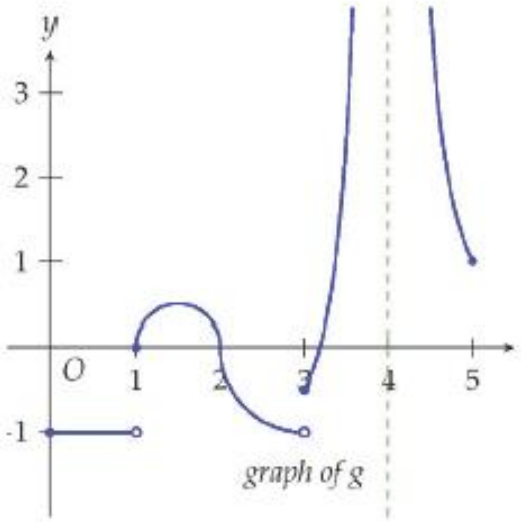
The graph of discontinuous function can’t be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.





Examples for continuous function

The graph of a function g is shown.

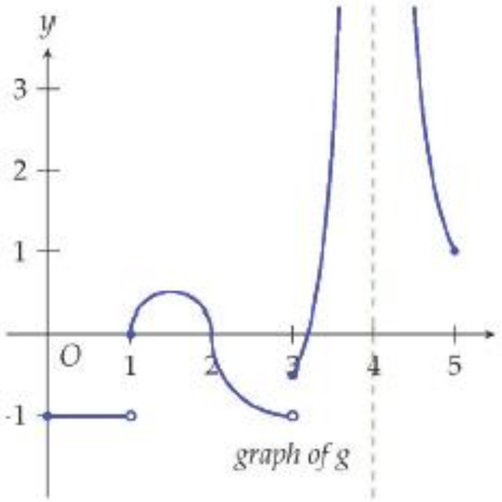


(a) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous?

(b) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the right?

(c) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the left?

Examples for continuous function



Solution:

(a) The function g is continuous at $a = 0, 2, 5$. In fact,

(i) The function g is continuous at $a = 0$, since $\lim_{x \to 0^+} g(x) = g(0) = -1$.

(ii) The function g is not continuous at $a = 1$, since $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$.

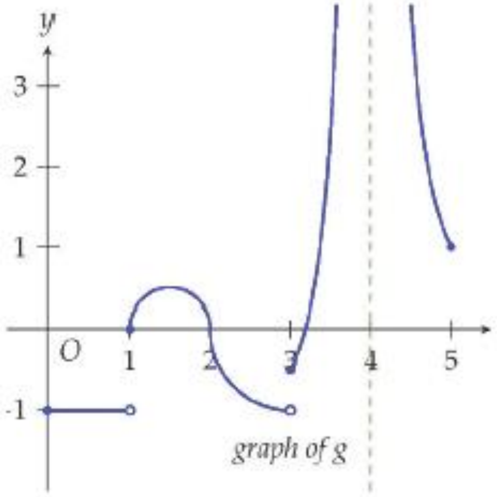
(iii) The function g is continuous at $a = 2$, since $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2) = 0$.

(iv) The function g is not continuous at $a = 3$, since $\lim_{x \to 3^-} g(x) \neq \lim_{x \to 3^+} g(x)$.

(v) The function g is not continuous at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is continuous at $a = 5$, since $\lim_{x \to 5^-} g(x) = g(5) = 1$.

Examples for continuous function



(b) The function g is continuous from the right at $a = 0, 1, 2, 3$. In fact,

(i) The function g is continuous from the right at $a = 0$, since $\lim_{x \to 0+} g(x) = g(0)$.

(ii) The function g is continuous from the right at $a = 1$, since $\lim_{x \to 1+} g(x) = g(1)$.

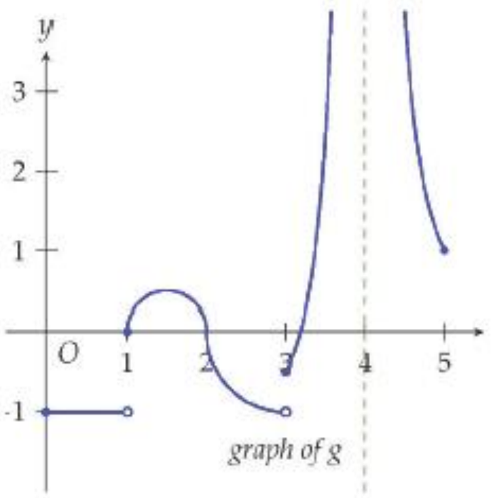
(iii) The function g is continuous from the right at $a = 2$, since $\lim_{x \to 2+} g(x) = g(2)$.

(iv) The function g is continuous from the right at $a = 3$, since $\lim_{x \to 3+} g(x) = g(3)$.

(v) The function g is not continuous from the right at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is not continuous from the right at $a = 5$, since $\lim_{x \to 5^+} g(x)$ does not exist.

Examples for continuous function



(c) The function g is continuous from the left at $a = 2, 5$. In fact,

(i) The function g is not continuous from the left at $a = 0$, since $\lim_{x \to 0^-} g(x)$ does not exist.

(ii) The function g is not continuous from the left at $a = 1$, since $\lim_{x \to 1^-} g(x) \neq g(1)$.

(iii) The function g is continuous from the left at $a = 2$, since $\lim_{x \to 2^-} g(x) = g(2)$.

(iv) The function g is not continuous from the left at $a = 3$, since $\lim_{x \to 3^-} g(x) \neq g(3)$.

(v) The function g is not continuous from the left at $a = 4$, since $g(4)$ does not exist.

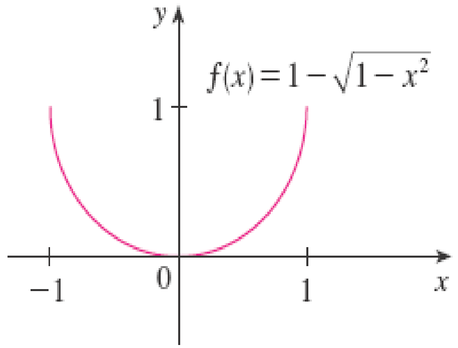
(vi) The function g is continuous from the left at $a = 5$, since $\lim_{x \to 5^-} g(x) = g(5)$.

Examples for continuous function

1) Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution:

If $-1 < a < 1$, then using the Limit Laws,



$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(1 - \sqrt{1 - x^2}\right)\\ &= 1 - \lim_{x\to a}\sqrt{1 - x^2}\\ &= 1 - \sqrt{\lim_{x\to a}(1 - x^2)}\\ &= 1 - \sqrt{1 - a^2}\\ &= f(a)\end{aligned}$$

f is continuous at a if $-1 < a < 1$.

$$\lim_{x\to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1.

$\therefore$ f is continuous on $[-1, 1]$.

Examples for continuous function

2) Show that the function $g(x)=2\sqrt{3-x}$ is continuous in $(-\infty,3]$

Solution: For $a < 3$, we have

$$\begin{aligned}
\lim_{x\to a} g(x) &= \lim_{x\to a} 2\sqrt{3-x} \\
&= 2\lim_{x\to a}\sqrt{3-x} \\
&= 2\sqrt{\lim_{x\to a}(3-x)} \\
&= 2\sqrt{\lim_{x\to a}3-\lim_{x\to a}x} \\
&= 2\sqrt{3-a} \\
&= g(a)
\end{aligned}$$

So g is continuous at $x=a$ for every a in $(-\infty,3)$.

Also, $\lim_{x\to 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3.

Thus, g is continuous on $(-\infty,3]$.

Examples for continuous function

3) Show that the function $f(x)=\dfrac{2x+3}{x-2}$ is continuous in $(2,\infty)$

Solution: For $a > 2$, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{2x+3}{x-2} = \frac{\lim\limits_{x\to a}(2x+3)}{\lim\limits_{x\to a}(x-2)}$$

$$= \frac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2}$$

$$= \frac{2a+3}{a-2}$$

$$= f(a)$$

Thus, f is continuous at $x = a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

Examples for continuous function

4) Show that the function $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$ is continuous in (-∞,∞)

Solution:

We know that trigonometric functions are continuous.

Therefore sin *x* is continuous in (-∞, π/4) and cos *x* is continuous in (π/4, ∞).

Thus *f* (x) is continuous in (-∞, π/4) ∪ (π/4, ∞).

$$\lim_{x \to (\pi/4)^-} f(x) = \lim_{x \to (\pi/4)^-} \sin x = \sin \frac{\pi}{4} = 1/\sqrt{2}$$

$$\lim_{x \to (\pi/4)^+} f(x) = \lim_{x \to (\pi/4)^+} \cos x = 1/\sqrt{2}$$

Thus, $\lim_{x \to (\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$,

Also, *f* (π/4) = $1/\sqrt{2}$, Therefore, f is continuous at $\pi/4$,

so f is continuous on $(-\infty, \infty)$.

Examples for continuous function

5) Use continuity to evaluate $\lim_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

Solution:

Take $f(x) = \arcsin x$ and $g(x) = \dfrac{1-\sqrt{x}}{1-x} \Rightarrow f(g(x)) = \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

Because arcsin is a continuous function, apply result (vi) $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

$$\begin{aligned}
\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) &= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right) \\
&= \arcsin\frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

Practice problems:

Use continuity to evaluate (i) $\lim_{x\to\pi} \sin(x + \sin x)$ (ii) $\lim_{x\to 1} e^{x^2-x}$

Examples for continuous function

6) Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

Solution:

(a) Take $f(x) = \sin x$ and $g(x) = x^2 \Rightarrow f(g(x)) = \sin(x^2) = h(x)$.

Since $f(x)$ is trigonometric function then it is continuous everywhere.

Since g(x) is polynomial then it is also continuous on R.

Thus by result (vii) h(x) = $f(g(x))$ is also continuous on R.

(b) Take $f(x) = \text{In}\, x$ and $g(x) = 1 + \cos x \Rightarrow f(g(x)) = \text{In}\,(1 + \cos x) = F(x)$.

Since $f(x)$ is logarithmic function then it is continuous for $x > 0$.

Since g(x) is the sum of constant and trigonometric functions then it is also continuous.

Thus by result (vii) F(x) = $f(g(x))$ is also continuous when (1+cosx) > 0.

i.e. F(x) is not defined when 1+cos x = 0 and this happens when x = ± π , = ± 3π, ….

Thus F(x) has discontinuities when x is an odd multiples of π and it is continuous on the intervals between these values.

Examples for continuous function

Continuous function
Results on cont. funs.

7) Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

Solution:

Since logarithmic functions and inverse trigonometric functions are continuous in their domain then the function In x is continuous for $x > 0$ and $\tan^{-1} x$ is continuous on $(-\infty, \infty)$

Thus, by result (i) In $x + \tan^{-1} x$ is continuous on $(0,\infty)$.

The denominator x^2-1 is a polynomial, so it is continuous everywhere. Therefore, by result (v)

f is continuous at all positive numbers x except $x = 1$.

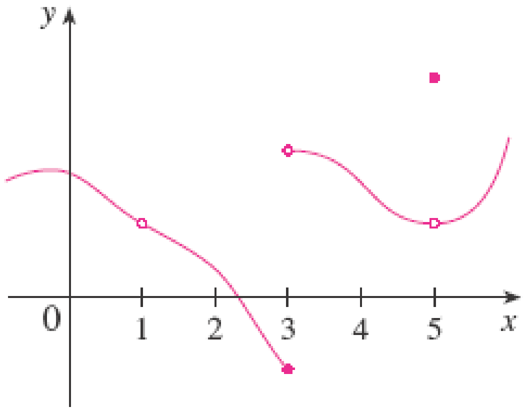
So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$.

Practice problem:

Show that $f(x) = \dfrac{7x^5 + x - 2}{x^2 - 4}$ is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

Examples for discontinuous function

8) At which numbers is f discontinuous? Why?



Solution:

The above graph has a hole (break) at $x = 1$ and $f(1)$ is not defined. So f is discontinuous at 1.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same).

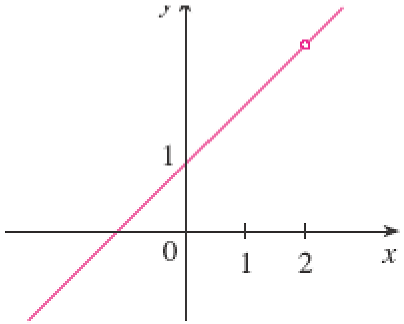
But $\lim_{x\to 5} f(x) \neq f(5)$

So f is discontinuous at 5.

Examples for discontinuous function

9) Where are each of the following functions discontinuous?

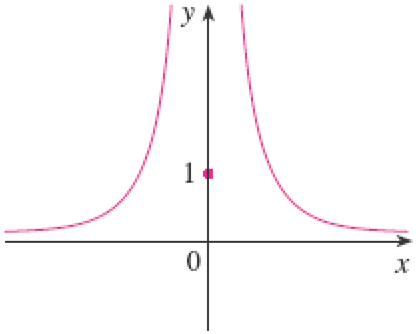
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



Solution:

Notice that $f(2)$ is not defined, so f is discontinuous at 2.

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



Solution:

Here $f(0) = 1$ is defined but

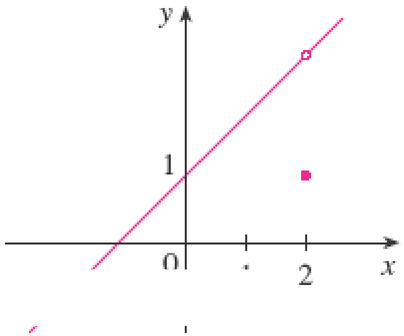
$\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{1}{x^2}$ does not exist. $\because \lim_{x\to 0}(1/x^2) = \infty$ So f is discontinuous at 0.

Note:

The kind of discontinuity discussed in (b) is called **infinite discontinuity** because it takes the limit value as infinity.

Examples for discontinuous function

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



Solution:

Here $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+1) = 3 \quad \text{exists.}$$

But $\lim_{x\to 2} f(x) \neq f(2)$

so f is not continuous at 2.

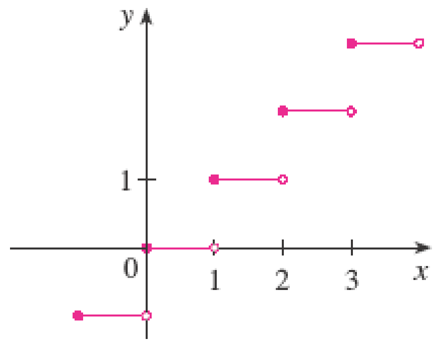
Note:
The kind of discontinuity discussed in (c) is called **removable discontinuity** because it could be removed by redefining $f(x) = 3$ at $x = 2$.

Examples for discontinuous function

(d) $f(x) = [\![x]\!]$

The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x.

$[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.



Solution:

At each integer n, the function $f(x) = [\![x]\!]$ is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but $$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$

$\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer. $\therefore f(x) = [\![x]\!]$ has discontinuities at all of the integers

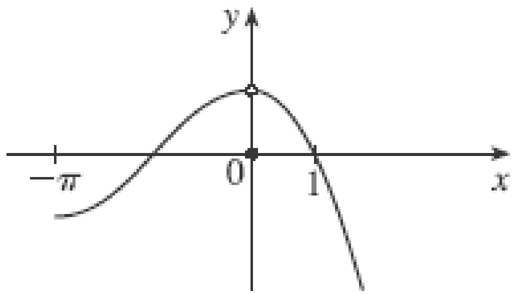
Note:
The kind of discontinuity discussed in (d) is called **jump discontinuity** because the function "jumps" from one value to another.

Definition: The absolute difference between the right hand limit and the left hand limit at a finite discontinuous point is called **jump of the function** at the discontinuous point.

Examples for discontinuous function

(e) $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$

Solution:



Given $f(0) = 0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos x = \cos 0 = 1.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 - x^2) = 1 - 0 = 1.$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \Rightarrow \lim_{x \to 0} f(x) \text{ exist which is } = 1.$$

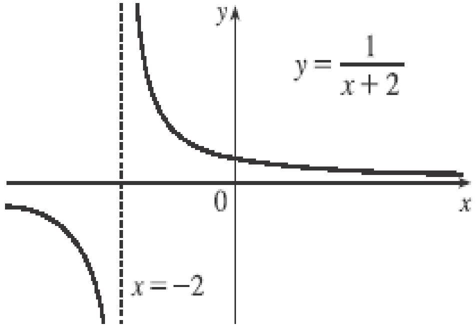
But $\lim_{x \to 0} f(x) \neq f(0)$.

$\therefore f$ is discontinuous at $x = 0$.

Examples for discontinuous function

10) **Practice problems:** Discuss the discontinuity of the following functions

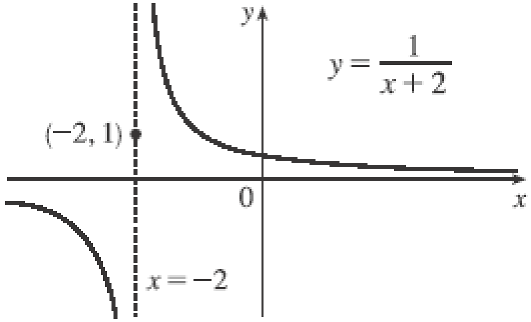
(i) $f(x) = \dfrac{1}{x+2}$ (Hint: proceed as problem (a))



Solution:

$f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.

(ii) $f(x) = \begin{cases} \dfrac{1}{x+2}, & x \neq -2 \\ 1, & x = -2 \end{cases}$ (Hint: proceed as problem (b))



Solution:

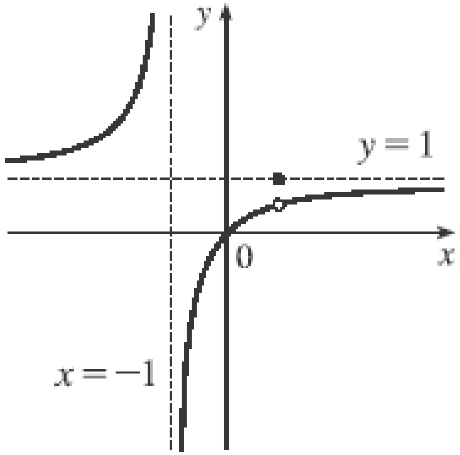
Here $f(-2) = 1$, but $\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = \infty$,

so $\lim_{x \to -2} f(x)$ does not exist and f is discontinuous at -2.

Examples for discontinuous function

$$(iii)\ f(x)=\begin{cases}\dfrac{x^2-x}{x^2-1}, & x\neq 1\\ 1, & x=1\end{cases}$$ (Hint: proceed as problem (c))

Solution:



$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-x}{x^2-1}=\lim_{x\to 1}\frac{x(x-1)}{(x+1)(x-1)}=\lim_{x\to 1}\frac{x}{x+1}=\frac{1}{2},$$

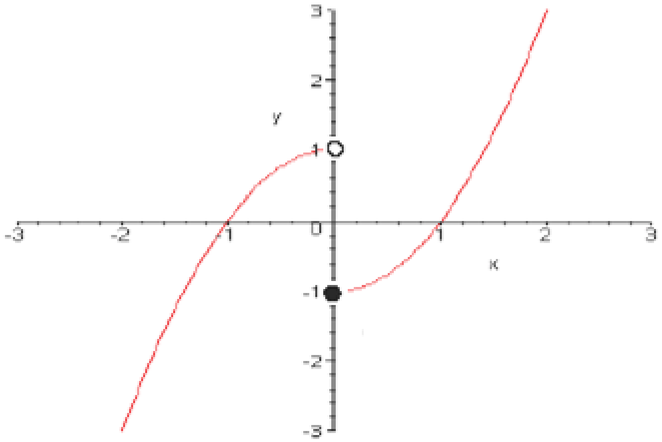
but $f(1)=1$, so f is discontinous at 1.

Examples for discontinuous function

11) Let the function f(x) be defined for all values of x by $f(x)=\begin{cases} x^2-1, & for \quad x\geq 0 \\ -x^2+1, & for \quad x<0 \end{cases}$

Draw the graph of the function f(x) and test the continuity from the graph.

Solution.



Here, $f(0)=-1 \Rightarrow f$ is defined at $x=0$.

$\lim\limits_{x\to 0^-} f(x)=1$ and $\lim\limits_{x\to 0^+} f(x)=-1 \Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist.

$\therefore f(x)$ is discontinuous at $x=0$.

Examples for discontinuous function

12) For the following f(x), find the points at which f(x) is discontinuous. Also, find the points at which f(x) is continuous from right and from the left. Justify your answer. $f(x)=\begin{cases} 1+x; & x\le 1 \\ 1/x; & 1<x<3 \\ \sqrt{x-3}; & x\ge 3 \end{cases}$

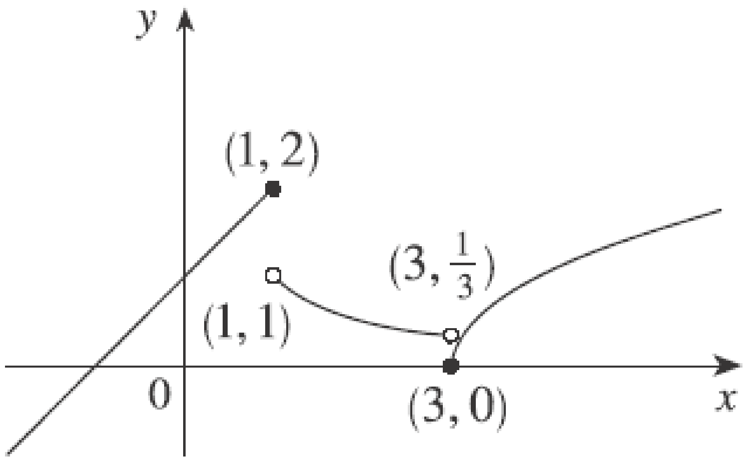
Solution:

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1} 1/x=1$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 1+x=2$

$\lim_{x\to 1^+} f(x)\ne \lim_{x\to 1^-} f(x) \Rightarrow$ At x = 1, the function is discontinuous

$\lim_{x\to 3^+} f(x)=\lim_{x\to 3}\sqrt{x-3}=0$ $f(x)=\begin{cases} 1+x; & x\le 1 \\ 1/x; & 1<x<3 \\ \sqrt{x-3}; & x\ge 3 \end{cases}$



$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 1/x=\frac{1}{3}$

$\lim_{x\to 3^+} f(x)\ne \lim_{x\to 3^-} f(x) \Rightarrow$ At x = 3, the function is discontinuous

$\lim_{x\to 3^+} f(x)=0=f(3) \Rightarrow$ At x = 3, the function is continuous from right.

$\lim_{x\to 1^-} f(x)=2=f(1) \Rightarrow$ At x = 1, the function is continuous from left.

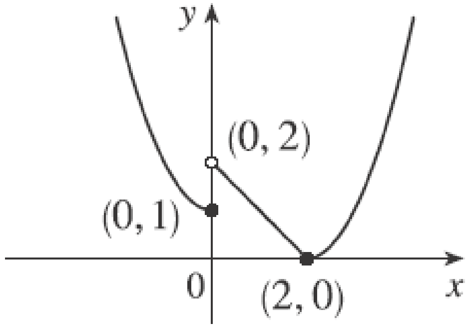
Examples for discontinuous function

13) Find the domain of where the following function f is continuous. Also find the numbers at which the function f is discontinuous.

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \leqslant 0 \\ 2 - x & \text{if } 0 < x \leqslant 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

Solution:

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$



f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (1 + x^2) = 1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (2 - x) = 0$, $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

$\therefore$ The domain of continuity is $(-\infty,0) \cup (0,\infty)$.

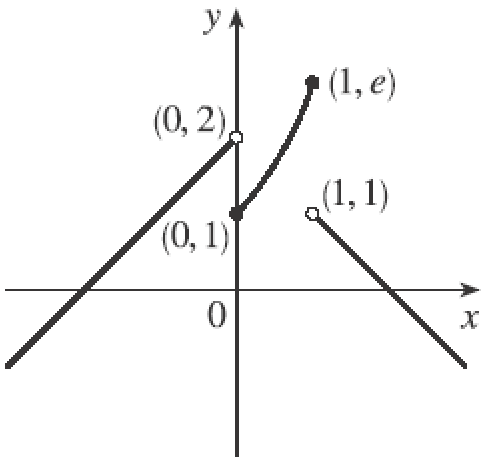
Examples for discontinuous function

14) **Practice problems:**

For the following f(x), find the points at which f(x) is discontinuous. Also, find the points at which f(x) is continuous from right and from the left. Justify your answer. $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x>1 \end{cases}$

Solution:

f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomi



it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = 2$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x = 1$,

so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0.

Also $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} e^x = e$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2-x) = 1$, so f is discontinuous at 1.

Since $f(1) = e$, f is continuous from the left at 1.

Examples for discontinuous function

15) For what value of the constant "c" is the function "f" continuous on (-∞, ∞),

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Solution:

$cx^2 + 2x$ & $x^3 - cx$ are polynomials in x $\Rightarrow$ They are continous everywhere in (-∞, ∞).

The only point to be checked is x = 2.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} x^3 - cx = 2^3 - c(2) = 8 - 2c$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} cx^2 + 2x = c2^2 + 2(2) = 4c + 4$$

If "f" is continuous at x = 2 , then $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

$$\Rightarrow 8 - 2c = 4c + 4 \Rightarrow 6c = 4 \Rightarrow c = \frac{2}{3}$$

Examples for discontinuous function

16) Find the values of ‘a’ and ‘b’ such that the function $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if} \quad x < 2 \\ ax^2 - bx + 3 & \text{if} \quad 2 \le x < 3 \\ 2x - a + b & \text{if} \quad x \ge 3 \end{cases}$

is continuous.

Solution:

At $x = 2$: $\displaystyle \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2^-} (x+2) = 2+2 = 4$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$$

We must have $4a - 2b + 3 = 4$, or $\mathbf{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\displaystyle \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x - a + b) = 6 - a + b$$

We must have $9a - 3b + 3 = 6 - a + b$, or $\mathbf{10a - 4b = 3}$ **(2)**.

Examples for discontinuous function

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rcr} -8a + 4b &=& -2 \\ 10a - 4b &=& 3 \\ \hline 2a \qquad &=& 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$

Thus, for f to be continuous $a = b = \frac{1}{2}$.

Note:

The above problem may also be asked in the following manner

Find the values of 'a' and 'b' such that the function $f(x) = \begin{cases} 2 + x; & x < 2 \\ ax^2 - bx + 3; & 2 \le x \le 3 \\ 2x - a + b; & x \ge 3 \end{cases}$

is continuous everywhere.

Derivative of a function

DERIVATIVE OF A FUNCTION

The derivative of a function represents the rate of change of a variable with respect to another variable. For example, the velocity of a body is defined as the rate of change of the location of the body with respect to time. The location is the *dependent* variable while time is the *independent* variable. Now if we measure the rate of change of velocity with respect to time, we get the acceleration of the body. In this case, the velocity is the *dependent* variable while time is the *independent* variable.

The **derivative** of a function $f(x)$ at $x=a$ is defined as $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Note:

1. If f is differentiable at a, then f is continuous at a.
2. The converse is not always true. (i.e) A function can be continuous at a, but not differentiable at a.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative,

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$$

Derivative of a function

17) Find $f'(3)$ if $f(x)=4x^2$ using definition of the derivative.

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$=\lim_{h\to 0}\frac{4(3+h)^2-4(3)^2}{h}$$

$$=\lim_{h\to 0}\frac{4(9+h^2+6h)-36}{h}$$

$$=\lim_{h\to 0}\frac{36+4h^2+24h-36}{h}$$

$$=\lim_{h\to 0}\frac{h(4h+24)}{h}$$

$$=\lim_{h\to 0}(4h+24)=24$$

$$=24$$

Derivative of a function

18) Find $f'\left(\frac{\pi}{4}\right)$ if $f(x)=sin(2x)$ using the first principle of derivative.

Solution: $$f'\left(\frac{\pi}{4}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{4}+h\right)-f\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{sin\left(2\left(\frac{\pi}{4}+h\right)\right)-sin\left(2\left(\frac{\pi}{4}\right)\right)}{h}$$

$$=\lim_{h\to 0}\frac{sin\left(\frac{\pi}{2}+2h\right)-sin\left(\frac{\pi}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{sin\left(\frac{\pi}{2}\right)cos(2h)+cos\left(\frac{\pi}{2}\right)sin(2h)-sin\left(\frac{\pi}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{cos(2h)+0-1}{h}$$

$$=\lim_{h\to 0}\frac{cos(2h)-1}{h}$$

=0

Derivative of a function

19) Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h\to 0} (2a + h - 8) \quad = 2a - 8$$

Theorem: If f is differentiable at a, then f is continuous at a.

Note: The converse of the above theorem is not true.

Derivative of a function

20) Find the domain at which the function f(x) = |x| is continuous and differentiable.

Solution: We know that, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Since *x* and –*x* are polynomial then $|x|$ is continuous in the interval (-∞,0) and (0,∞).

Now check the continuity at *x* = 0.

For $x > 0$, $|x| = x$, $\therefore \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

For $x < 0$, $|x| = -x$, $\therefore \lim_{x \to 0^-} |x| = \lim_{x \to 0^+} (-x) = 0$

$\therefore \lim_{x \to 0^+} |x| = \lim_{x \to 0^-} |x| = \lim_{x \to 0} |x| = 0$

$\Rightarrow$ at $x = 0$, f(x) is continuous. Therefore, the domain of the continuity of f(x) is (-∞, ∞)

To check the differentiability

If $x > 0$ then $|x| = x$ and $|x + h| = x + h$.

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\Rightarrow$ f(x) is differentiable for $x > 0$

Derivative of a function

If $x < 0$ then $|x| = -x$ and $|x+h| = -(x+h)$.

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|x+h|-|x|}{h} = \lim_{h\to 0}\frac{-(x+h)-[-(x)]}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$$

$\Rightarrow$ f(x) is differentiable for $x<0$

If $x = 0$ then $|x| = 0$

$$\therefore\ f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

Now $\displaystyle\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$

and $\displaystyle\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$

$\displaystyle\lim_{h\to 0^+}\frac{|h|}{h} \neq \lim_{h\to 0^-}\frac{|h|}{h} \quad \Rightarrow \lim_{h\to 0}\frac{|h|}{h}$ does not exist. $\Rightarrow f'(0)$ does not exist.

$\Rightarrow$f(x) is differentiable for $x>0$ and $x<0$ but not differentiable at $x=0$.

Therefore, the domain of the differentiability of f(x) is $(-\infty, 0)\cup(0, \infty)$

DIFFERENTIATION FORMULA

1) $\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$

2) $\frac{d}{dx}\left(\text{constant}\right)=0$

3) $\frac{d}{dx}(\sin x)=\cos x$

4) $\frac{d}{dx}(\sin nx)=n\cos nx$

5) $\frac{d}{dx}\left(\cos x\right)=-\sin x$

6) $\frac{d}{dx}\left(\tan x\right)=\sec^{2} x$

7) $\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$

8) $\frac{d}{dx}\left(\cot x\right)=-\text{cosec}^{2} x$

9) $\frac{d}{dx}\left(\cos ec\, x\right)=-\text{co}\sec x\cot x$

10) $\frac{d}{dx}\left(\text{In}\, x\right)=\frac{d}{dx}\left(\log_{e} x\right)=\frac{1}{x}$

11) $\frac{d}{dx}\left(\log_{a} x\right)=\frac{1}{x}\frac{1}{\log_{e} a}$

12) $\frac{d}{dx}\left(a^{x}\right)=a^{x}\log a$

13) $\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$

14) $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^{2}}}$

15) $\frac{d}{dx}\left(\cos^{-1} x\right)=\frac{-1}{\sqrt{1-x^{2}}}$

16) $\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^{2}}$

17) $\frac{d}{dx}\left(\cot^{-1} x\right)=\frac{-1}{1+x^{2}}$

18) $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{x^{2}-1}}$

19) $\frac{d}{dx}\left(\text{cose}\,\text{c}^{-1} x\right)=\frac{-1}{x\sqrt{x^{2}-1}}$

20) $\frac{d}{dx}\left(\sinh x\right)=\cosh x$

1. **Find the derivative of $f(x)=10x^6$.**

Solution: $f(x)=10x^6$ $f'(x)=\frac{d}{dx}(10x^6)=10\frac{d}{dx}x^6=10(6x^5)=60x^5$

Derivatives of logarithmic and exponential functions

For the function of the type $f(x)=a^x$, where a is a constant, then $f'(x)=a^x \ln a$.

Derivative of $f(x)=e^x$ is $f'(x)=e^x$

For the logarithmic function $f(x)=\ln x$, $f'(x)=\frac{1}{x}$.

Derivative of $f(x)=\log_a x$ is $f'(x)=\frac{1}{x\ln a}$.

2. Find $\frac{dy}{dx}$. a. $f(x)=3x^{20}+10^{x}$ b. $f(x)=e^{x}+e^{5x}$

Solution.

a. $\frac{df}{dx}=60x^{19}+10^{x}\ln 10$, $b.\frac{df}{dx}=e^{x}+5e^{5x}$

3. Find $\frac{dy}{dx}$. a. $f(x)=3\ln x+10$ b. $f(x)=\log x+3$

Solution. a. $\frac{df}{dx}=\frac{3}{x}$, $b.\frac{df}{dx}=\frac{1}{x\ln 10}$

4. **Find the derivative of** $f(x)=3x^{3}+8$**.**

Solution: $f(x)=3x^{3}+8$

$$f'(x)=\frac{d}{dx}(3x^{3}+8)\ =\frac{d}{dx}(3x^{3})+\frac{d}{dx}(8)\ =3\frac{d}{dx}(x^{3})+0\quad =3(3x^{2})\ =9x^{2}$$

5. Differentiate the functions (addition/subtraction).

a. $f(x) = x^2 + 2x - 1$ b. $f(x) = 3x^{20} - 5x$ c. $f(x) = \dfrac{3}{\sqrt[3]{x}} + \sqrt{x}$

Solution.

a. $f'(x) = 2x + 2$

b. $f'(x) = 60x^{19} - 5$

c. $f'(x) = -\dfrac{1}{x^{4/3}} + \dfrac{1}{2\sqrt{x}}$

Product Rule: If $f(x) = u(x)v(x)$, then $f'(x) = u(x)\dfrac{d}{dx}v(x) + v(x)\dfrac{d}{dx}u(x)$.

6. **Find the derivative of** $f(x)=(2x^2-6)(3x^3+8)$

Solution : Using the product rule,

$f(x)=(2x^2-6)(3x^3+8)$

Let $u(x)=2x^2-6$ **and** $v(x)=3x^3+8$

Taking the derivative of $u(x)$,

$$\frac{du}{dx}=\frac{d}{dx}(2x^2-6)\ =\frac{d}{dx}(2x^2)-\frac{d}{dx}(6)\ =2\frac{d}{dx}(x^2)-0\ =2(2x)=4x$$

Taking the derivative of $v(x)$,

$$\frac{dv}{dx}=\frac{d}{dx}(3x^3+8)=\frac{d}{dx}(3x^3)+\frac{d}{dx}(8)=3\frac{d}{dx}(x^3)+0=3(3x^2)=9x^2$$

Using the formula for the product rule

$$f'(x) = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x) = (2x^2 - 6)(9x^2) + (3x^3 + 8)(4x)$$

$$= 18x^4 - 54x^2 + 12x^4 + 32x \ = 30x^4 - 54x^2 + 32x$$

7. Differentiate the functions a. $f(x) = (x^2 + 2x - 1)x^{10}$ b. $f(x) = (3x^{20} - 2)(5x)$

Solution.

a. $f'(x) = (2x + 2)x^{10} + (x^2 + 2x - 1)10x^9$

b. $f'(x) = (60x^{19})5x + (3x^{20} - 2)5$

Quotient Rule:

If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x)\frac{d}{dx}u(x) - u(x)\frac{d}{dx}v(x)}{(v(x))^2}$

8. **Find the derivative of** $f(x)=\dfrac{(2x^2-6)}{(3x^3+8)}$.

Solution:

Using the Quotient Rule, $f(x)=\dfrac{(2x^2-6)}{(3x^3+8)}$

Let $u(x)=2x^2-6$ **and** $v(x)=3x^3+8$

Taking the derivative of $u(x)$,

$$\frac{du}{dx}=\frac{d}{dx}(2x^2-6)=\frac{d}{dx}(2x^2)-\frac{d}{dx}(6)=2\frac{d}{dx}(x^2)-0=2(2x)=4x$$

Taking the derivative of $v(x)$,

$$\frac{dv}{dx}=\frac{d}{dx}(3x^3+8)=\frac{d}{dx}(3x^3)+\frac{d}{dx}(8)=3\frac{d}{dx}(x^3)+0=3(3x^2)=9x^2$$

Using the formula for the quotient rule, $f'(x) = \dfrac{v(x)\dfrac{d}{dx}u(x) - u(x)\dfrac{d}{dx}v(x)}{(v(x))^2}$

$$f'(x) = \frac{(3x^3+8)(4x)-(2x^2-6)(9x^2)}{(3x^3+8)^2} = \frac{12x^4+32x-18x^4+54x^2}{9x^6+48x^3+64} = \frac{-6x^4+54x^2+32x}{9x^6+48x^3+64}$$

9. Differentiate the function $f(x) = \dfrac{3x^2-5}{\sqrt{x}}$

Solution:

$$f'(x) = \frac{\sqrt{x}(6x)-(3x^2-5)\dfrac{1}{2\sqrt{x}}}{x}$$

The chain rule:

The chain rule is $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

10. Find $\dfrac{dy}{dx}$ by chain rule.

a. $y = u^{10}$ *and* $u = 1 - x^5$ b. $f(x) = (3x^{20} - 2)^{21}$ c. $f(x) = \sqrt{x^2 + 1}$

Solution.

a. $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 10u^9 \cdot (-5x^4) = -50x^4(1 - x^5)^9$

b. $\dfrac{dy}{dx} = 21(3x^{20} - 2)^{20}(60x^{19}) = 1260x^{19}(3x^{20} - 2)^{20}$

c. $\dfrac{dy}{dx} = \dfrac{2x}{2\sqrt{x^2 + 1}}$

11. Find $f'(x)$ if $f(x) = ln(3x^2 - 2x)$

Solution: By chain rule. $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{d(\ln u)}{du} \cdot \dfrac{du}{dx}$ $\qquad u = 3x^2 - 2x$

$$\frac{dy}{dx} = \frac{d(\ln u)}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{d(3x^2 - 2x)}{dx} = \frac{1}{3x^2 - 2x} \cdot (6x - 2)$$

12. If $f(x) = \sin(\cos(\tan x))$, *find* $f'(x)$

Solution:

$$f'(x) = \frac{d\left(\sin(\cos(\tan x))\right)}{dx}$$

$$f'(x) = \cos(\cos(\tan x))\frac{d(\cos(\tan x))}{dx}$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)]\frac{d(\tan x)}{dx}$$

$$= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x$$

13. Differentiate $y = e^{\sec 3\theta}$

Solution:

$$\frac{dy}{d\theta} = \frac{d\left(e^{\sec 3\theta}\right)}{d\theta} = e^{\sec 3\theta}\frac{d(\sec 3\theta)}{d\theta}$$

$$= e^{\sec 3\theta}\sec 3\theta\tan 3\theta\frac{d(3\theta)}{d\theta}$$

$$= e^{\sec 3\theta}\sec 3\theta\tan 3\theta(3) = 3e^{\sec 3\theta}\sec 3\theta\tan 3\theta$$

Implicit differentiation:

Functions like $3y^2 - 5xy + 9xy^5 - 2 = 0$, where x can not be written as independent variable with respect to another variable y. In this case direct differentiation is either difficult or impossible. To find derivative we will use implicit differentiation using chain rule. Following examples will illustrate the situation.

14. Find $\frac{dy}{dx}$ *or* y'.

a. $x^2 + y^2 = 2xy$ b. $3y^2 - 5xy + 9x - 2 = 0$ b. $y^2 = \log x + 3$

Solution:

a. $x^2 + y^2 = 2xy$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2xy)$$

DIFFERENTIAL CALCULUS

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(2xy)$$

$$2x+2y\frac{dy}{dx}=2x\frac{dy}{dx}+2y$$

$$2y\frac{dy}{dx}-2x\frac{dy}{dx}=2y-2x$$

$$(2y-2x)\frac{dy}{dx}=2y-2x$$

$$\frac{dy}{dx}=\frac{2y-2x}{2y-2x}$$

$$\frac{dy}{dx}=1$$

b. $6yy' - 5y - 5xy' + 9 = 0 \Rightarrow y' = \dfrac{5y-9}{6y-5x}$

c. $2yy' = \dfrac{1}{x\ln 10} \Rightarrow y' = \dfrac{1}{2xy\ln 10}$

15. **If $x^2 - xy + y^2 = 5$, find the value of y' and y''.**

Solution: $x^2 - xy + y^2 = 5$

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

DIFFERENTIAL CALCULUS

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$(-x + 2y)\frac{dy}{dx} = -2x + y$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

$$y' = \frac{y - 2x}{2y - x}$$

$$(2y - x)y' = y - 2x$$

$$\frac{d}{dx}((2y - x)y') = \frac{d}{dx}(y - 2x)$$

$$(2y-x)\frac{d}{dx}(y')+y'\frac{d}{dx}(2y-x)=\frac{d}{dx}(y)-\frac{d}{dx}(2x)$$

$$y''(2y-x)+y'(2y'-1)=y'-2$$

$$y''=\frac{2y'-2-2y'^2}{2y-x}$$

After substitution of y' ,

$$y''=\frac{2\dfrac{y-2x}{2y-x}-2-2\left(\dfrac{y-2x}{2y-x}\right)^2}{2y-x}=-\frac{6(y^2-xy+x^2)}{(2y-x)^3}$$

16. **Find $\frac{dy}{dx}$ if $\sin(x+y) = y^2 \cos x$**

Solution: Given $\sin(x+y) = y^2 \cos x$

Diff. w.r.to x, we get,

$$\cos(x+y)\left(1+y'\right) = y^2(-\sin x) + (\cos x)\left(2yy'\right)$$

$$\cos(x+y) + \cos(x+y)y' = -y^2 \sin x + 2yy' \cos x$$

$$\cos(x+y) + y^2 \sin x = \left(2y\cos x - \cos(x+y)\right)y'$$

$$\therefore y' = \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}$$

17. **If $x^4 + y^4 = 16$, find the value of y' and y''.**

Solution: $x^4 + y^4 = 16 \qquad 4x^3 + 4y^3 y' = 0 \Rightarrow y' = -\dfrac{4x^3}{4y^3} = -\dfrac{x^3}{y^3}$

Differentiating with respect to x

$$y'' = -\left[\frac{y^3(3x^2) - x^3(3y^2 y')}{(y^3)^2}\right] = -\left[\frac{3y^2x^2\{y - xy'\}}{y^6}\right]$$

$$y' = -\frac{x^3}{y^3} \Rightarrow y'' = -\left[\frac{3y^2x^2\left\{y - x\left[-\frac{x^3}{y^3}\right]\right\}}{y^6}\right] = -\left[\frac{3x^2\left\{\frac{y(y^3) + x(x^3)}{y^3}\right\}}{y^4}\right]$$

$$= -\left[3x^2\left\{\frac{y^4 + x^4}{y^{3+4}}\right\}\right] = -\left[3x^2\left\{\frac{16}{y^7}\right\}\right] \qquad \because y^4 + x^4 = 16$$

$$= -\frac{48x^2}{y^7}$$

18. Find the slope of the tangent line to the curve $3xy-2x^2=7$ at (1, 3). Also compute the second derivative.

Solution.

$$3xy'+3y-4x=0 \Rightarrow y'=\frac{4x-3y}{3x}=\frac{4(1)-3(3)}{3(1)}=-\frac{5}{3}$$

For the second derivative we consider again $3xy'+3y-4x=0$

Taking derivative we find $3xy''+3y'+3y'-4=0 \Rightarrow y''=\dfrac{4-6y'}{3x}=\dfrac{6y-4x}{3x^2}=\dfrac{14}{3}$

TANGENT LINE

19. Find the tangent line to the equation $x^3 + y^3 = 6xy$ at the point (3, 3) and at what point the tangent line is horizontal in the first quadrant.

Solution:

Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get $3x^2 + 3y^2 y' = 6xy' + 6y$

$$x^2 + y^2 y' = 2xy' + 2y$$

$$y^2 y' - 2xy' = 2y - x^2$$

$$y' (y^2 - 2x) = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

DIFFERENTIAL CALCULUS

At (3, 3), $y' = \dfrac{2y - x^2}{y^2 - 2x} = \dfrac{2(3) - 3^2}{3^2 - 2(3)} = -1$

Slope ofthe tangent is -1

Equation of the tangent is given by $y - y_1 = m\left(x - x_1\right)$

$\Rightarrow y - 3 = (-1)\left(x - 3\right) \Rightarrow y - 3 = -x + 3 \Rightarrow x + y = 6$

The tangent line is horizontal if $y' = 0$

$\Rightarrow \dfrac{2y - x^2}{y^2 - 2x} = 0 \Rightarrow 2y - x^2 = 0 \Rightarrow 2y = x^2 \Rightarrow y = \dfrac{x^2}{2}$ provided $y^2 - 2x \neq 0$

$y = \dfrac{x^2}{2}$ *in* $x^3 + y^3 = 6xy$

$$\Rightarrow x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$\Rightarrow x^3 + \frac{x^6}{8} = 3x^3$$

$$\Rightarrow \frac{8x^3 + x^6}{8} = 3x^3$$

$$\Rightarrow 8x^3 + x^6 = 24x^3$$

$$\Rightarrow 8x^3 - 24x^3 = -x^6$$

$$\Rightarrow -16x^3 = -x^6$$

$$\Rightarrow x^6 - 16x^3 = 0 \Rightarrow x^3(x^3 - 16) = 0 \Rightarrow x = 0,\ (16)^{\frac{1}{3}} \Rightarrow x = 0,\ (2^4)^{\frac{1}{3}} \Rightarrow x = 0,\ (2)^{\frac{4}{3}}$$

$$y = \frac{x^2}{2} = \frac{\left(2^{\frac{4}{3}}\right)^2}{2} = \frac{\left(2^{\frac{8}{3}}\right)}{2} = 2^{\frac{8}{3}-1} = 2^{\frac{5}{3}}$$

Thus the tangent line is horizontal at $(2^{\frac{4}{3}}, 2^{\frac{5}{3}})$

Maximum and Minimum values of f

The maximum and minimum values of f are called extreme values of f.

Definition:

Let f be defined on $[a,b]$. f is said to have an absolute maximum (or global maximum) on $[a,b]$ if there is at least one point $c \in [a,b]$ such that $f(x) \leq f(c), \forall x \in [a,b]$.

i.e., the largest value of f on $[a,b]$ is called the **absolute maximum**.

Definition:

Let f be defined on $[a,b]$. f is said to have an absolute minimum (or global minimum) on $[a,b]$ if there is at least one point $c \in [a,b]$ such that $f(x) \geq f(c), \forall x \in [a,b]$.

i.e., the least value of f on $[a,b]$ is called the **absolute minimum**.

Definition:

Let f be the function defined on $[a,b]$ and let $c \in (a,b)$, then

(i). f is said to have a **local maximum** (or relative maximum) at c, if there is a neighbourhood $(c-\delta, c+\delta)$ of c such that $f(x) < f(c), \forall x \in (c-\delta, c+\delta)$, $x \neq c$.

i.e., $f(c)$ is the greatest value in a neighbourhood of c.

(ii). f is said to have a **local minimum** (or relative minimum) at c, if there is a neighbourhood $(c-\delta, c+\delta)$ of c such that $f(x) > f(c), \forall x \in (c-\delta, c+\delta)$, $x \neq c$.

i.e., $f(c)$ is the least value in a neighbourhood of c.

Definition: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$.

1. Find the values of f at the critical numbers of f in (a,b) (the open interval)

2. Find the values of f at the endpoints of the interval, $f(a)$ and $f(b)$.

3. The largest value from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

1. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1, -\frac{1}{2} \le x \le 4$.

Solution:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x$$

$f'(x) = 0$

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0 \;\; \Rightarrow x = 0, 2$

The critical points are $x = 0, 2$

$f(0) = 0^3 - 3(0)^2 + 1 = 1$

$f(2) = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$

Also, at the end points

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 1 = -\frac{1}{8} - 3\left(\frac{1}{4}\right) + 1 = \frac{-1-6+8}{8} = \frac{1}{8}$$

$f(4) = 4^3 - 3(4)^2 + 1 = 64 - 48 + 1 = 17$

The absolute minimum value is $f(2) = -3$ and it occurs at $x = 2$,

and the absolute maximum value is $f(4) = 17$ and it occurs at $x = 4$.

2. Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval $f(x)=x^3-6x^2+5,[-3,5]$.

Solution:

$$f(x)=x^3-6x^2+5$$

$$f'(x)=3x^2-12x$$

$$f'(x)=0$$

$$\Rightarrow 3(x^2-4x)=0$$

$$\Rightarrow x(x-4)=0$$

$$\Rightarrow x=0,4$$

The critical points are $x=0,4$

$$f(0)=0^3-6(0^2)+5=5$$

$$f(4)=4^3-6(4^2)+5=-27$$

Also, at the end points

$f(-3)=(-3)^3-6(-3)^2+5=-76$

$f(5)=5^3-6(5^2)+5=-20$

Hence, the absolute minimum value is $f(-3)=-76$ and it occurs at $x=-3$

and the absolute maximum value is $f(0)=5$ and it occurs at $x=0$.

3. Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval $f(x)=(x^2-4)^3,[-2,3]$.

Solution:

$f(x)=(x^2-4)^3$

$f'(x)=3(x^2-4)^2(2x)=6x(x^2-4)^2$

$f'(x)=0$

$\Rightarrow 6x(x^2-4)^2=0$

$\Rightarrow x = 0, \left(x^2 - 4\right)^2 = 0$

$\Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm\ 2$

The critical points are $x = 0, 2, -2$

$f(0) = (0-4)^3 = -64$

$f(2) = \left(2^2 - 4\right)^3 = 0$

$f(-2) = \left((-2)^2 - 4\right)^3 = 0$

Also, at the end points

$f(-2) = \left((-2)^2 - 4\right)^3 = 0$

$f(3) = \left(3^2 - 4\right)^3 = 125$

The absolute minimum value is $f(0) = -64$ and it occurs at $x = 0$

and the absolute maximum value is $f(3) = 125$ and it occurs at $x = 3$.

4. Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval $f(x)=2\cos x-\sin 2x, \left[0,\frac{\pi}{2}\right]$.

Solution:

$$f(x)=2\cos x-\sin 2x$$

$$f'(x)=-2\sin x-2\cos 2x$$

$$f'(x)=0$$

$$\Rightarrow -2(\sin x+\cos 2x)=0$$

$$\sin x+\cos 2x=0$$

$\Rightarrow x=\frac{\pi}{2}$ is the critical point

$$f\left(\frac{\pi}{2}\right)=2\cos\left(\frac{\pi}{2}\right)-\sin 2\left(\frac{\pi}{2}\right)=0$$

Also at the end points

$f(0) = 2\cos 0 - \sin 0 = 2$

$f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right) - \sin 2\left(\frac{\pi}{2}\right) = 0$

The absolute minimum value is $f\left(\frac{\pi}{2}\right) = 0$ and it occurs at $x = \frac{\pi}{2}$

and the absolute maximum value is $f(0) = 2$ and it occurs at $x = 0$.

5. Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval $f(x) = x - 2\sin x, [0, 2\pi]$.

Solution:

$f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x$

$$f'(x)=0$$

$$\Rightarrow 1-2\cos x=0$$

$$\Rightarrow \cos x=\frac{1}{2}$$

$\Rightarrow x=\dfrac{\pi}{3},\dfrac{5\pi}{3}$ are the critical point

$$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-2\sin\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-2\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\sqrt{3}=-0.684853$$

$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}-2\sin\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}-2\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{3}+\sqrt{3}=6.968039$$

Also at the end points

$$f(0)=0-2\sin 0=0$$

$$f(2\pi)=2\pi-2\sin(2\pi)=2\pi-0=6.28$$

The absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.684853$ and it occurs at $x = \frac{\pi}{3}$

and the absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.968039$ and it occurs at $x = \frac{5\pi}{3}$.

6. Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval

$f(x) = x - \log x, \left[\frac{1}{2}, 2\right]$.

Solution:

$$f(x) = x - \log x$$

$$f'(x) = 1 - \frac{1}{x}$$

$$f'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0$$

$$\Rightarrow \frac{x-1}{x} = 0 \Rightarrow x = 1$$

and also $f'(x)$ does not exist at $\Rightarrow x = 0$

$\therefore x = 0, 1$ are the critical points

But $x = 0$ does not belong to $\left[\frac{1}{2}, 2\right]$

$$f(1) = 1 - \log 1 = 1 - 0 = 1$$

Also, at the end points

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - \log\frac{1}{2} = 0.5 - (-0.6931) = 1.1931$$

$$f(2) = 2 - \log 2 = 2 - 0.6931 = 1.3068$$

The absolute minimum value is $f(1) = 1$ and it occurs at $x = 1$

and the absolute maximum value is $f(2) = 1.3068$ and it occurs at $x = 2$.

7. Find the absolute maximum and minimum values for $f(t)=t\sqrt{4-t^2}$ in the interval $[-1,2]$.

Solution:

$$f(t)=t\sqrt{4-t^2}$$

$$f'(t)=\sqrt{4-t^2}\,(1)+t\left(\frac{1}{2}\right)\left(4-t^2\right)^{\frac{1}{2}-1}(-2t)$$

$$=\sqrt{4-t^2}-t^2\left(4-t^2\right)^{-\frac{1}{2}}$$

$$=\sqrt{4-t^2}-t^2(4-t^2)^{-\frac{1}{2}}$$

$$=\sqrt{4-t^2}-\frac{t^2}{\sqrt{4-t^2}}$$

$$=\frac{4-t^2-t^2}{\sqrt{4-t^2}}$$

$$f'(t)=\frac{4-2t^2}{\sqrt{4-t^2}}$$

To find the critical points Put $f'(t)=0 \Rightarrow \dfrac{4-2t^2}{\sqrt{4-t^2}}=0$

$\Rightarrow 4-2t^2=0 \Rightarrow t^2=2 \Rightarrow t=\pm\sqrt{2}$

and also $f'(t)$ does not exist at $4-t^2=0 \Rightarrow t^2=4 \Rightarrow t=\pm 2$

$t=2,-2,\sqrt{2},-\sqrt{2}$ are the critical points

But $t=-2,-\sqrt{2}$ does not belong to $[-1,2]$

$f(2)=(2)\sqrt{4-(2)^2}=0$

$f(\sqrt{2})=(\sqrt{2})\sqrt{4-(\sqrt{2})^2}=2$

Also, at the end points

$f(-1)=(-1)\sqrt{4-(-1)^2}=-\sqrt{3}$

The absolute maximum value is $f(\sqrt{2})=2$ and it occurs at $t=\sqrt{2}$

The absolute minimum value is $f(-1)=-\sqrt{3}$ and it occurs at $t=-1$.

Increasing/Decreasing Test:

(a) If $f'(x) > 0$ on an interval, then *f* is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then *f* is decreasing on that interval.

Definition:

If the graph of *f* lies above all of its tangents on an interval *I*, then it is called concave upward on *I*.

If the graph of *f* lies below all of its tangents on an interval *I*, then it is called concave downward on *I*.

Concavity Test:

Suppose that a function f is differentiable on an open interval containing c and that $f''(c)$ exists. Then

1. If $f''(c) > 0$, then the graph of *f* is concave upward at the point $P(c, f(c))$.

2. If $f''(c) < 0$, then the graph of *f* is concave downward at the point $P(c, f(c))$.

Definition: Inflection Point (Point of Inflection or Flex Point):

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

(i.e) A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and if the concavity of the curve changes there (from upward to downward, or from downward to upward).

Note:

1. The possible points for inflection points are points where either $f'' = 0$ or f'' is undefined.
2. However, just like that not every critical point is a local max / min, not every such point is an inflection point either.
3. They are just the pool of points you need to check in order to find the inflection point(s) of a curve.
4. It often happens that a graph has different concavity on the two sides of a vertical asymptote.
5. However, because a curve is not continuous at a vertical asymptote, it can never have an inflection point there, even if f is defined there.

The First Derivative Test:

Suppose that c is a critical number of a continuous function f.

(a). If f' changes from positive to negative at c, then f has a local maximum at c.

(b). If f' changes from negative to positive at c, then f has a local minimum at c.

(c). If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

The Second Derivative Test:

Suppose $f''(c)$ is continuous near c. If f is differentiable on an open interval containing c, then

1. If $f'(c)=0$ and $f''(c)<0$, then f has local maximum at $x=c$.

2. If $f'(c)=0$ and $f''(c)>0$, then f has local minimum at $x=c$.

8. For the function $f(x) = 2x^3 - 3x^2 - 12x$ find the maximum and minimum of $f(x)$. Also find the intervals on which $f(x)$ is increasing and decreasing.

Solution:

Domain of the function $f(x) = 2x^3 - 3x^2 - 12x$ is $(-\infty, \infty)$

$$f'(x) = 6x^2 - 6x - 12$$

$$f''(x) = 12x - 6$$

To find critical points,

$$f'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$\Rightarrow x = -1, 2$ are the critical points

By First Derivative Test:

Interval	$f'(x)=6x^2-6x-12$	$f(x)=2x^3-3x^2-12x$	Conclusion
$-\infty<x<-1$	Positive	Increasing on $(-\infty,-1)$	$f'(x)$ changes positive to negative at $x=-1$ $\Rightarrow$ Local maximum at $x=-1$
$-1<x<2$	Negative	Decreasing on $(-1,2)$	
$2<x<\infty$	Positive	Increasing on $(2,\infty)$	$f'(x)$ changes negative to positive at $x=2$ $\Rightarrow$ Local maximum at $x=2$

Local Maximum Value is $=f(-1)=2(-1)^3-3(-1)^2-12(-1)=-2-3+12=7$

Local Minimum Value is $=f(2)=2(2)^3-3(2)^2-12(2)=16-12-24=-20$.

9. Find the maximum and minimum values of $f(x)=\dfrac{x^2+x+1}{x^2-x+1}$.

Solution:

$$f(x)=\frac{x^2+x+1}{x^2-x+1}$$

$$f'(x)=\frac{\left(x^2-x+1\right)(2x+1)-\left(x^2+x+1\right)(2x-1)}{\left(x^2-x+1\right)^2}$$

$$=\frac{\left(2x^3-2x^2+2x+x^2-x+1\right)-\left(2x^3+2x^2+2x-x^2-x-1\right)}{\left(x^2-x+1\right)^2}$$

$$=\frac{\left(-x^2+1\right)-\left(x^2-1\right)}{\left(x^2-x+1\right)^2}=\frac{-2x^2+2}{\left(x^2-x+1\right)^2}=\frac{2(1-x^2)}{\left(x^2-x+1\right)^2}$$

$$f''(x)=\frac{\left(x^2-x+1\right)^2\left(2(0-2x)\right)-2\left(1-x^2\right)2\left(x^2-x+1\right)(2x-1)}{\left(x^2-x+1\right)^4}$$

$$f''(x)=\frac{(x^2-x+1)\left[(-4x)(x^2-x+1)-4(1-x^2)(2x-1)\right]}{(x^2-x+1)^4}$$

$$=\frac{\left[-4x^3+4x^2-4x-4(2x-2x^3-1+x^2)\right]}{(x^2-x+1)^3}$$

$$=\frac{\left[-4x^3+4x^2-4x-8x+8x^3+4-4x^2\right]}{(x^2-x+1)^3}=\frac{\left[4x^3-12x+4\right]}{(x^2-x+1)^3}$$

To find the critical points $f'(x)=0\Rightarrow\dfrac{2(1-x^2)}{(x^2-x+1)^2}=0\Rightarrow 2(1-x^2)=0\Rightarrow x=\pm1$

And also $f'(x)$ is not defined when the denominator is 0 (i.e) $(x^2-x+1)^2=0\Rightarrow x^2-x+1=0$

By applying the formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\dfrac{1\pm\sqrt{-3}}{2}=\dfrac{1\pm i\sqrt{3}}{2}$

Which are complex numbers

Therefore $x = \pm 1$ are the critical points

By Second derivative test

$$f''(1) = \frac{\left[4(1)^3 - 12(1) + 4\right]}{\left((1)^2 - (1) + 1\right)^3} = -4 < 0$$

$\Rightarrow$ Local maximum attains at $x = 1$

$$f''(-1) = \frac{\left[4(-1)^3 - 12(-1) + 4\right]}{\left((-1)^2 - (-1) + 1\right)^3} = \frac{\left[-4 + 12 + 4\right]}{\left(1 + 1 + 1\right)^3} = \frac{12}{27} > 0$$

$\Rightarrow$ Local minimum attains at $x = -1$

Local Maximum Value is $f(1) = \dfrac{(1)^2 + (1) + 1}{(1)^2 - (1) + 1} = 3$

Local Minimum Value is $f(-1) = \dfrac{(-1)^2 + (-1) + 1}{(-1)^2 - (-1) + 1} = \dfrac{1}{3}$.

10. Find the minimum and maximum value of $f(x)=x^2-2x-5$ in the interval [0,5].

Solution:

$f(x)=x^2-2x-5$

$f'(x)=2x-2$

$f'(x)=0 \Rightarrow 2x-2=0 \Rightarrow x=1$

$\Rightarrow x=1$ is the critical point

$f(1)=(1)^2-2(1)-5=-6$

Also, at the end points

$f(0)=(0)^2-2(0)-5=-5$

$f(5)=(5)^2-2(5)-5=10$

Hence, the minimum value is $f(1)=-6$ occurs at $x=1$, and the maximum value is $f(5)=10$ occurs at $x=5$.

11. Find the local maximum and minimum values of the function $f(x) = x + 2\sin x,\ 0 \le x \le 2\pi$.

Solution:

$$f(x) = x + 2\sin x$$

$$f'(x) = 1 + 2\cos x$$

To find the critical points

Put $f'(x) = 0$

$$\Rightarrow 1 + 2\cos x = 0$$

$$\Rightarrow \cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3} \text{ and } x = \frac{4\pi}{3}$$

By First Derivative test:

Interval	$f'(x)=1+2\cos x$	$f(x)=x+2\sin x$	Conclusion
$0<x<\frac{2\pi}{3}$	Positive	Increasing on $\left(0,\frac{2\pi}{3}\right)$	$f'(x)$ changes from positive to negative at $\frac{2\pi}{3}$. $\Rightarrow$ Local maximum at $x=\frac{2\pi}{3}$
$\frac{2\pi}{3}<x<\frac{4\pi}{3}$	Negative	Decreasing on $\left(\frac{2\pi}{3},\frac{4\pi}{3}\right)$	
$\frac{4\pi}{3}<x<2\pi$	Positive	Increasing on $\left(\frac{4\pi}{3},2\pi\right)$	$f'(x)$ changes from negative to positive at $\frac{4\pi}{3}$. $\Rightarrow$ Local minimum at $x=\frac{4\pi}{3}$

Local maximum Value is $= f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\frac{\sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$

Local Minimum Value is $= f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$

12. Find the local maximum and minimum values of the function $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both the first and second derivative tests.

Solution:

Domain of the function $f(x) = \sqrt{x} - \sqrt[4]{x}$ is $(0, \infty)$

$$f(x) = x^{\frac{1}{2}} - x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{2}x^{\left(\frac{1}{2}\right)-1} - \frac{1}{4}x^{\left(\frac{1}{4}\right)-1} = \frac{1}{2}x^{-\left(\frac{1}{2}\right)} - \frac{1}{4}x^{-\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2x^{\left(\frac{1}{2}\right)}} - \frac{1}{4x^{\left(\frac{3}{4}\right)}} = \frac{1}{2x^{\left(\frac{1}{2}\right)}}\left(1 - \frac{1}{2x^{\left(\frac{3}{4}\right)-\left(\frac{1}{2}\right)}}\right) = \frac{1}{2x^{\left(\frac{1}{2}\right)}}\left(1 - \frac{1}{2x^{\left(\frac{1}{4}\right)}}\right)$$

$$= \frac{1}{2x^{\left(\frac{1}{2}\right)}}\left(\frac{2x^{\left(\frac{1}{4}\right)}-1}{2x^{\left(\frac{1}{4}\right)}}\right) = \frac{2x^{\left(\frac{1}{4}\right)}-1}{4x^{\left(\frac{3}{4}\right)}} = \frac{2\sqrt[4]{x}-1}{4\sqrt[4]{x^3}}$$

To find the critical points

$$f'(x) = 0,\ \frac{2\sqrt[4]{x}-1}{4\sqrt[4]{x^3}} = 0$$

$$\Rightarrow 2\sqrt[4]{x} - 1 = 0$$

$$\Rightarrow \sqrt[4]{x} = \frac{1}{2}$$

$$\Rightarrow x = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

and also $f'(x)$ does not exist at $x = 0$

Therefore, the critical points are $x = \frac{1}{16}$ & $x = 0$

By First Derivative Test:

Interval	$f'(x)=\dfrac{2\sqrt[4]{x}-1}{4\sqrt[4]{x^3}}$	$f(x)=\sqrt{x}-\sqrt[4]{x}$	Conclusion
$0<x<\dfrac{1}{16}$	Negative	Decreasing on $\left(0,\dfrac{1}{16}\right)$	$f'(x)$ changes from negative to positive at $\dfrac{1}{16}$. $\Rightarrow$ Local minimum at $x=\dfrac{1}{16}$
$\dfrac{1}{16}<x<\infty$	Positive	Increasing on $\left(\dfrac{1}{16},\infty\right)$	

Local Minimum Value is $=f\left(\dfrac{1}{16}\right)=\sqrt{\dfrac{1}{16}}-\sqrt[4]{\dfrac{1}{16}}=\dfrac{1}{4}-\dfrac{1}{2}=-\dfrac{1}{4}$.

By Second Derivative test:

$$f''(x)=\frac{d}{dx}\left(f'(x)\right)=\frac{d}{dx}\left[\frac{1}{2}x^{-\left(\frac{1}{2}\right)}-\frac{1}{4}x^{-\left(\frac{3}{4}\right)}\right]$$

$$=-\frac{1}{2}\frac{1}{2}x^{-\left(\frac{1}{2}\right)-1}+\frac{1}{4}\frac{3}{4}x^{-\left(\frac{3}{4}\right)-1}$$

$$=\frac{1}{4}x^{-\left(\frac{3}{2}\right)}+\frac{3}{16}x^{-\left(\frac{7}{4}\right)}$$

$$f''(0)=\frac{1}{4}(0)^{-\left(\frac{3}{2}\right)}+\frac{3}{16}(0)^{-\left(\frac{7}{4}\right)}=0$$

$\Rightarrow x=0$ is neither local maximum or local minimum

$\Rightarrow x=0$ is the inflection point.

$$f''\left(\frac{1}{16}\right)=\frac{1}{4}\left(\frac{1}{16}\right)^{-\left(\frac{3}{2}\right)}+\frac{3}{16}\left(\frac{1}{16}\right)^{-\left(\frac{7}{4}\right)}=\frac{1}{4}\left(\frac{16}{1}\right)^{\left(\frac{3}{2}\right)}+\frac{3}{16}\left(\frac{16}{1}\right)^{\left(\frac{7}{4}\right)}$$

$$=\frac{1}{4}\left[(16)^{\left(\frac{1}{2}\right)}\right]^{3}+\frac{3}{16}\left[(16)^{\left(\frac{1}{4}\right)}\right]^{7}=\frac{1}{4}[4]^{3}+\frac{3}{16}[2]^{7}>0$$

$\Rightarrow$ Local minimum attains at $x=\frac{1}{16}$

Local Minimum Value is $=f\left(\frac{1}{16}\right)=\sqrt{\frac{1}{16}}-\sqrt[4]{\frac{1}{16}}=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$.

13. If $f(x)=2x^3+3x^2-36x$ find the intervals on which is increasing or decreasing, the local maxima and minima, the intervals of concavity and the points of inflection?

Solution:

Domain of the function $f(x)=2x^3+3x^2-36x$ is $(-\infty,\infty)$

$f'(x)=6x^2+6x-36$

$f''(x)=12x+6$

To find critical points $f'(x)=0 \Rightarrow 6x^2+6x-36=0$

$\Rightarrow 6(x+3)(x-2)=0$

$x=-3,2$ are the critical points

By First derivative test

Interval	$f'(x)=6x^2+6x-36$	$f(x)=2x^3+3x^2-36x$	Conclusion
$-\infty<x<-3$	Positive	Increasing on $(-\infty,-3)$	$f'(x)$ changes from positive to negative at $x=-3$ $\Rightarrow$ Local maximum at $x=-3$
$-3<x<2$	Negative	Decreasing on $(-3,2)$	$f'(x)$ changes from negative to positive at $x=2$ $\Rightarrow$ Local minimum at $x=2$
$2<x<\infty$	Positive	Increasing on $(2,\infty)$	

Local maximum value is $f(-3)=2(-3)^3+3(-3)^2-36(-3)=81$

Local minimum value is $f(2)=2(2)^3+3(2)^2-36(2)=-44$

Concavity Test:

To find inflection points, $f''(x)=0 \Rightarrow 12x+6=0$

$\Rightarrow x=-\dfrac{1}{2}$ is the inflection points

Interval	$f''(x)=12x+6$	$f(x)=2x^3+3x^2-36x$
$-\infty < x < -\dfrac{1}{2}$	Negative	Concave downward on $\left(-\infty,-\dfrac{1}{2}\right)$
$-\dfrac{1}{2} < x < \infty$	Positive	Concave upward on $\left(-\dfrac{1}{2},\infty\right)$

Since the curve changes from concave downward to upward at $x=-\dfrac{1}{2}$.

Thus the point of inflection is $\left(-\dfrac{1}{2}, f\left(-\dfrac{1}{2}\right)\right)$

$$f\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2-36\left(-\frac{1}{2}\right)$$

$$=2\left(-\frac{1}{8}\right)+3\left(\frac{1}{4}\right)+18$$

$$=-\frac{1}{4}+\frac{3}{4}+18$$

$$f\left(-\frac{1}{2}\right)=\frac{37}{2}$$

$\therefore$ The Point of Inflection is $\left(-\frac{1}{2},\frac{37}{2}\right)$

14. Given $f(x) = x^3 - 12x + 2$, (i). Find the intervals of increase or decrease, (ii). Find the local maximum and minimum values, (iii). Find the intervals of concavity and the inflection points.

Solution:

Domain of the function $f(x) = x^3 - 12x + 2$ is $(-\infty, \infty)$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

To find critical points $f'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\Rightarrow x = 2, -2$ are the critical points.

By First Derivative Test:

Interval	$f'(x)=3x^2-12$	$f(x)=x^3-12x+2$	Conclusion
$-\infty<x<-2$	Positive	Increasing on $(-\infty,-2)$	$f'(x)$ changes from positive to negative at $x=-2$ $\Rightarrow$ Local maximum at $x=-2$
$-2<x<2$	Negative	Decreasing on $(-2,2)$	
$2<x<\infty$	Positive	Increasing on $(2,\infty)$	$f'(x)$ changes from negative to positive at $x=2$ $\Rightarrow$ Local minimum at $x=2$

Local Maximum Value is $= f(-2)=(-2)^3-12(-2)+2=-8+24+2=18$

Local Minimum Value is $= f(2)=(2)^3-12(2)+2=8-24+2=-14$

Concavity test:

To find inflection points, $f''(x)=0 \Rightarrow 6x=0 \Rightarrow x=0$

$\Rightarrow x=0$ is the inflection points

Interval	$f''(x)=6x$	$f(x)=x^3-12x+2$
$-\infty<x<0$	Negative	Concave downward on $(-\infty,0)$
$0<x<\infty$	Positive	Concave upward on $(0,\infty)$

Since the curve changes from concave downward to upward at $x=0$

Thus the point of inflection is $(0, f(0))$

$f(0)=0^3-12.0+2=2$

$\therefore$ The Point of Inflection is $(0,2)$.

15. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution:

Domain of the function $y = f(x) = x^4 - 4x^3$ is $(-\infty, \infty)$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

To find critical points, $f'(x) = 0$

$\Rightarrow 4x^3 - 12x^2 = 0$

$\Rightarrow 4x^2(x-3) = 0$

$\Rightarrow x = 0, 3$ are the critical points

By second derivative test

$f''(0) = 12(0)^2 - 24(0) = 0$

$\Rightarrow x=0$ is neither local maximum or local minimum

$\Rightarrow x=0$ is the inflection point.

$f''(3)=12(3)^2-24(3)=36>0$

$\Rightarrow$ Local minimum attains at $x=3$

Local Minimum value is $f(3)=3^4-4(3^3)=81-108=-27$

Concavity Test:

To find inflection points, $f''(x)=0$

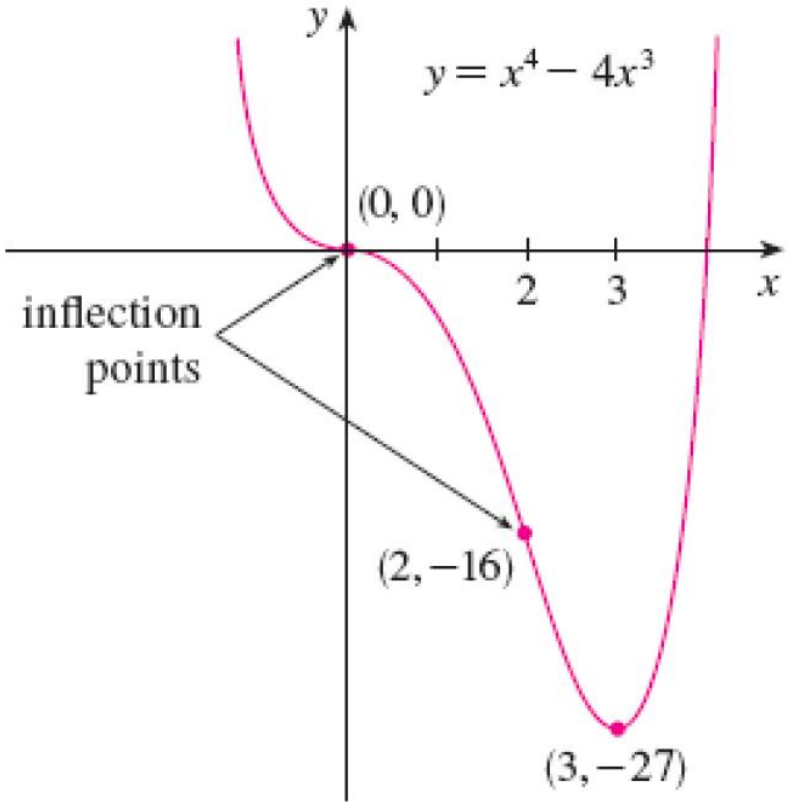
$\Rightarrow 12x^2-24x=0$

$\Rightarrow 12x(x-2)=0$

$\Rightarrow x=0,2$ are the inflection points

Interval	$y'' = 12x^2 - 24x = 12x(x-2)$	$y = x^4 - 4x^3$
$-\infty < x < 0$	Positive	Concave upward on $(-\infty, 0)$
$0 < x < 2$	Negative	Concave downward on $(0, 2)$
$2 < x < \infty$	Positive	Concave upward on $(2, \infty)$

Graph



16. APPLICATIONS OF DERIVATIVES

Problem.16 Find the dimensions of a circular cylinder whose volume is 9 m^3 but which uses the least amount of material.

Solution:

The total surface area, A of the cylinder is

A = top surface + side surface + bottom surface

$= \pi r^2 + 2\pi rh + \pi r^2$

$= 2\pi r^2 + 2\pi rh$

The volume, V of the cylinder is $V = \pi r^2 h$ since $V = 9\,m^3$

We can write $9 = \pi r^2 h \Rightarrow h = \dfrac{9}{\pi r^2}$

This gives the surface area just in terms of r as

$$A = 2\pi r^2 + 2\pi r\left(\frac{9}{\pi r^2}\right) = 2\pi r^2 + \frac{18}{r} = 2\pi r^2 + 18r^{-1}$$

To find the minimum, take the first derivative of A with respect to r as

$$\frac{dA}{dr} = 4\pi r + 18(-1)r^{-2} = 4\pi r - \frac{18}{r^2}$$

Solving for $\frac{dA}{dr} = 0$

$$4\pi r - \frac{18}{r^2} = 0 \Rightarrow 4\pi r^3 - 18 = 0$$

$$r^3 = \frac{18}{4\pi}$$

$$r = \left(\frac{18}{4\pi}\right)^{\frac{1}{3}} = 1.12725\,\text{m}$$

Since $h = \frac{9}{\pi r^2}$, $\Rightarrow h = \frac{9}{\pi(1.12725)^2} = 2.2545\text{m}$

$$\frac{d^2A}{dr^2} = 4\pi - 18(-2)r^{-3} = 4\pi + \frac{36}{r^3}$$

$$\left[\frac{d^2A}{dr^2}\right]_{r=1.12725} = 4\pi + \frac{36}{1.12725} = 44.5025$$

This value $\dfrac{d^2A}{dr^2} > 0$ for $r = 1.12725\,m$

As per the second derivative test, $r = 1.12725\,m$ corresponds to a minimum.

MA 8151 - ENGINEERING MATHEMATICS - I

UNIT - I

Differential Calculus.

1.1 Representation of functions:

Function:

A function f from a set D to a set E is a rule that assigns a unique element $f(x) \in E$ to each element $x \in D$.

The set D of all possible input values is called the domain of the function.

The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Real-valued functions:

A function whose domain and co-domain are subsets of the set of all real numbers is known as real-valued function.

Explicit functions:

If x and y be so related that y can be expressed explicity in terms x, then y is called explicit function of x.

Ex: $y = x^2 - 4x + 2$.

Implicit functions:

If x and y be so related that y cannot be expressed explicitly in terms of x, then y is called implicit function of x.

Eg: $x^3+y^3-3xy=0$.

Domain, co-domain, range and image:

Let $f: A \to B$ then

set A is called the domain of the function

set B is called co-domain

The set of all the images of all the elements of A under the function f is called the range of f and is denoted by $f(A)$.

The range of f is $f(A)=\{f(x): x \in A\}$.

Clearly $f(A) \subseteq B$.

If $x \in A$, $y \in B$ and $y=f(x)$, then y is called the image of x under f.

Graph of functions:

If f is a function with domain D, then its graph is the set of ordered pairs $\{(x, f(x)) / x \in D\}$.

Piecewise - defined functions:

The functions are described by using different formula's on different parts of its domain. Such functions are called piecewise-d.f.

The vertical line test for a function:

A curve in the xy-plane is the graph of a function of x iff no vertical line intersects the curve more than once.

Even function and odd function:

If a function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ and odd function of x if $f(-x) = -f(x)$ for every number x in its domain.

Problem based on the domain and range and sketch the graph of the function:

1. Find the domain and range and sketch the graph of the function $f(x) = x^2$

Solution: Given: $f(x) = x^2$

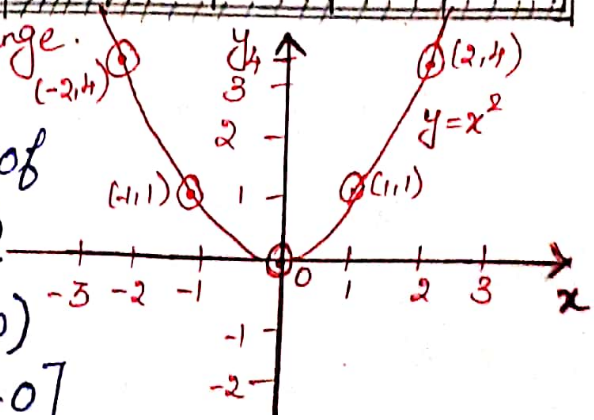
$\Rightarrow y = x^2.$

$\Rightarrow$ Given equation is a parabola. Domain

Domain(x)	$-\infty$	...	-2	-1	0	1	2	....	∞
Range (y)	∞	...	4	1	0	1	4	....	∞

Here $x, y \in \mathbb{R}$.

Range.



So the domain of f is the set of all real numbers $\mathbb{R}$. ie, $(-\infty, \infty)$.

The graph of the range is $[0, \infty)$

$[\because x^2 \geq 0]$

2. Find the domain and sketch the graph of the function $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x\geq 0. \end{cases}$

Solution:

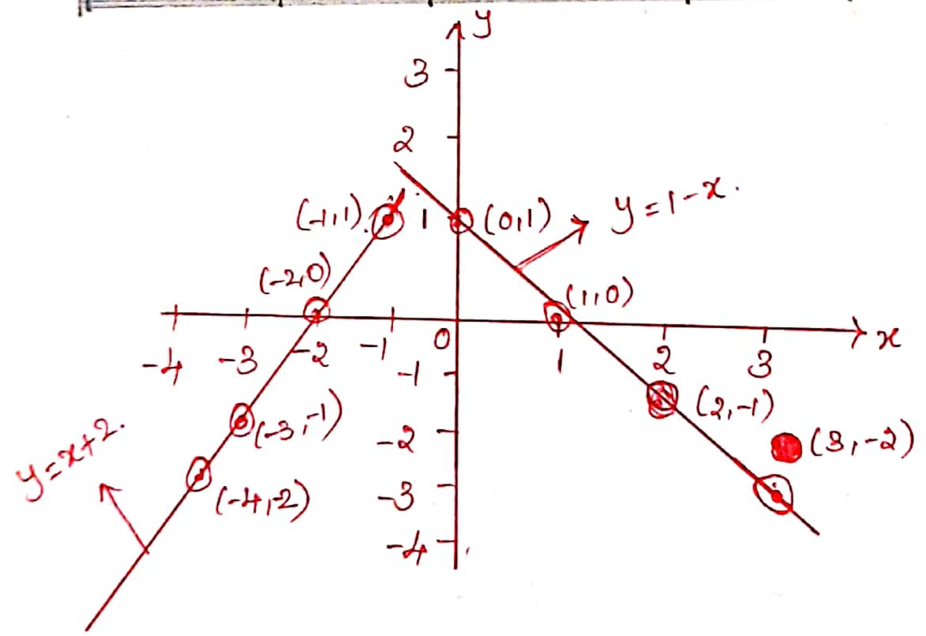
Given: $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x\geq 0. \end{cases}$

ie, $y = x+2$, $x<0$

$y = 1-x$, $x \geq 0$.

$x<0$	-1	-2	-3	-4	....
$y=x+2$	1	0	-1	-2	....

$x\geq 0$	0	1	2	3	...
$y=1-x$	1	0	-1	-2	...



$\therefore$ The domain is $(-\infty, \infty)$.

3. Find the domain and the range of each function $f(x) = \dfrac{4}{3-x}$.

Solution: Given : $f(x) = \frac{4}{3-x}$

$\Rightarrow y = \frac{4}{3-x}$ (divison by zero is not allowed)

Domain (x)	$-\infty$	...	-2	-1	0	1	2	...	3	...	4	...	∞
Range (y)	0	...	4/5	1	4/3	2	4	...∞	N.D	$-\infty$...	-4	...	0

Here $x, y \in R$, N.D → Not defined.

So, the domain is $(-\infty, 3) \cup (3, \infty)$ {3} - (or)

and the range is $(-\infty, 0) \cup (0, \infty)$.

4. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$.

Solution: Given: $f(x) = \frac{x+4}{x^2-9}$ {±3} - (or)

$\Rightarrow y = \frac{x+4}{x^2-9}$

$\Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$.

So the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

5. Find the domain of the function $f(x) = \sqrt{x+2}$.

Solution: Given : $f(x) = \sqrt{x+2}$

$\Rightarrow y = \sqrt{x+2}$.

$x + 2 \geq 0$ [$\because$ square root of a negative number is not defined]

$x \geq -2$.

So the domain is $[-2, \infty)$.

Even function : $f(-x) = f(x)$

Odd function : $f(-x) = -f(x)$.

* **Determine whether each of the function is even, odd or neither even nor odd.**

1. $f(x) = x^2+1$

Given : $f(x) = x^2+1$

$$f(-x) = (-x)^2+1$$
$$= x^2+1$$
$$f(-x) = f(x)$$

$\therefore$ $f(x)$ is an even function.

2. $f(x) = x \cos x$.

$\left[\because \cos(-\theta) = \cos\theta \quad \sin(-\theta) = -\sin\theta\right]$

Given: $f(x) = x\cos x$

$$f(-x) = (-x)\cos(-x)$$
$$= -x \cos x$$
$$f(-x) = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

3. $f(x) = x+1$

Given : $f(x) = x+1$

$$f(-x) = -x+1$$
$$f(-x) \neq f(x)$$
$$f(-x) = -(x-1)$$
$$f(-x) \neq -f(x)$$

$\therefore$ $f(x)$ is neither even nor odd function.

Evaluate the difference quotient for the given function

1. $f(x) = 4+3x-x^2$, $\dfrac{f(3+h)-f(3)}{h}$

Solution: Given: $f(x) = 4+3x-x^2$

$f(3+h) = 4+3(3+h)-(3+h)^2 = 4+9+3h-(9+h^2+6h)$

$= -h^2-3h+4$

$f(3) = 4+3(3)-(3)^2 = 4+9-9 = 4$

$\dfrac{f(3+h)-f(3)}{h} = \dfrac{-h^2-3h+4-4}{h} = \dfrac{-h^2-3h}{h} = -3-h.$

H.W

1. $f(x) = x^3$, $\dfrac{f(a+h)-f(a)}{h}$ Ans: $h^2+3ah+3a^2$

2. $f(x) = 2x^2-5x+1$, $\dfrac{f(a+h)-f(a)}{h}$, $h \neq 0$.

3. $f(x) = 3x^2-x+2$, find $f(a)$, $f(-a)$, $f(a+1)$, $f(a^2)$ and $f(a+h)$ Ans: i) $3a^2-a+2$ ii) $3a^2+a+2$ iii) $3a^2+5a+4$ iv) $3a^4-a^2+2$, v) $3a^2+6ah+3h^2-a-h+2$.

4. $f(x) = 1-x^4$, $f(x) = x^5+x$, Check it even or odd.

5. Sketch the graph and find the domain and range of the function $f(x) = 2x-1$.

6. Find the domain of the function $f(x) = \dfrac{1}{x^2-x}$.

$x^2-x=0.$

$x(x-1)=0$

$x=0, x=1$

$(-\infty,0)\ (0,1)\ (1,\infty).$

Definition:

$$\lim_{x\to a} f(x) = L \text{ iff } \lim_{x\to a^-} f(x) = L \text{ \& } \lim_{x\to a^+} f(x) = L.$$

Infinite Limits:

Let f be a function defined on both sides of a, except possibly at a itself.
Then $\lim_{x\to a} f(x) = \infty$ mean that $f(x)$ can be arbitrarily large by taking x sufficiently close to a, but not equal to a.

Definition: The line $x = a$ is called a vertical asymptote of the curve $f(x)$ if atleast one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \quad \lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty \quad \lim_{x\to a^-} f(x) = -\infty.$$

Determine the infinite Limit:

1. $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$

Solution: Given: $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$

$$= \frac{2-1}{(1-1)^2} = \frac{1}{0}$$

$$\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty.$$

2. $\lim_{x \to -3^+} \left(\frac{x+2}{x+3} \right)$

NOTE:
1) +ve, +ve = ∞
2) −ve, −ve = ∞
3) +ve, −ve = $-\infty$
4) −ve, +ve = $-\infty$
5) +ve = ∞
6) −ve = $-\infty$.

Solution: Given: $\lim_{x \to -3^+} \left(\frac{x+2}{x+3} \right)$

$x \to -3^+ \Rightarrow x$ is close to -3 but larger than -3.

Nr = $x+2$ becomes negative

Dr = $x+3$ becomes positive

Let $x = -2.9$

Nr = $(-2.9+2 = -0.9 = -ve)$

Dr = $(-2.9+3 = 0.1 = +ve)$

$\therefore \lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty$.

3. $\lim_{x \to (2\pi)^-} x \operatorname{cosec} x$.

Solution: Given: $\lim_{x \to (2\pi)^-} x \operatorname{cosec} x = \lim_{x \to (2\pi)^-} \frac{x}{\sin x}$

$x \to (2\pi)^- \Rightarrow x$ is close to 2π but smaller than 2π

Nr = x becomes positive

Dr = $\sin x$ becomes negative $[\because \sin 290^\circ = -ve]$

$\therefore \lim_{x \to (2\pi)^-} x \operatorname{cosec} x = -\infty$.

4. $\lim_{x \to 3^+} \log(x^2 - 9)$

Solution:

Given: $\lim_{x \to 3^+} \log(x^2 - 9)$

$x \to 3^+ \Rightarrow x$ is close to 3 but larger than 3

$\log(x^2 - 9)$ becomes negative $[\because \log[(3.1)^2 - 9] = -ve]$

$\therefore \lim_{x \to 3^+} \log(x^2 - 9) = -\infty$.

5. $\lim_{x \to 0^+} \left(\frac{1}{x} - \log x \right) = \infty$
(+ve)

6. $\lim_{x \to -3^-} \left(\frac{x+2}{x+3} \right) = \infty$
(−ve, −ve)

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7. Sketch the graph of the function

$$f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1. \end{cases}$$

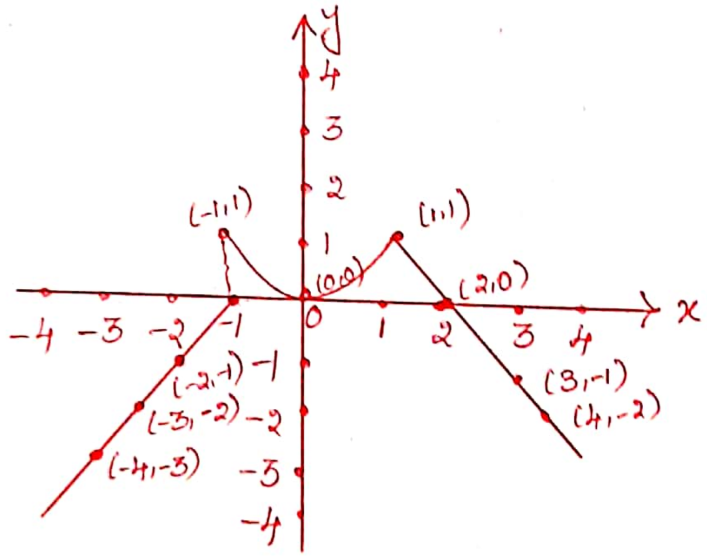
[Jan : 2018, 2017]

(6)

and use it to determine the values of 'a' for which $\lim_{x \to a} f(x)$ exists.

Solution:

	$1+x ; x < -1$			$x^2 ; -1 \le x \le 1$			$2-x^2, x \ge 1$			
x	-2	-3	-4	-1	0	1	1	2	3	4
$f(x)$	-1	-2	-3	1	0	1	1	0	-1	-2



At $x = -1$,

$$f(-1^-) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (1+x) = 1-1 = 0$$

$$f(-1) = \lim_{x \to -1} f(x) = \lim_{x \to -1} x^2 = (-1)^2 = 1$$

$$f(-1^+) = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 = (-1)^2 = 1$$

Here, $f(-1^-) \neq f(-1) = f(-1^+)$

$\therefore$ f is discontinuous at $x = -1$.

At $x=1$

$$f(1^-) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = (1)^2 = 1$$

$$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = (1)^2 = 1.$$

$$f(1^+) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1.$$

Here, $f(1^-) = f(1) = f(1^+)$:

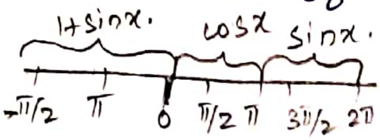
$\therefore$ f is continuous at $x=1$.

Hence, $\lim\limits_{x\to a} f(x)$ exists for all 'a' except at $a=-1$.

	0	$\pi/2$	π	$3\pi/2$	2π
$\sin x$:	0	1	0	-1	0
$\cos x$:	1	0	-1	0	1

H.W.

8. Sketch the graph of the function $f(x) = \begin{cases} 1+\sin x, & x<0 \\ \cos x, & 0\le x\le \pi \\ \sin x, & x>\pi \end{cases}$ and use it to determine the value of 'a' for which $\lim\limits_{x\to a} f(x)$ exists.



$\cos\pi = -1$
$\sin\pi = 0$
$\sin 0 = 0$
$\cos 0 = 1$

Eliminating Zero Denominators Algebraically

If the denominator is zero, cancelling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c.

1. Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution: Given: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1) = 1+1 = 2.$$

$$\therefore \lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

2. $\lim_{x \to 1} \frac{x^4-1}{x^3-1}$

$\left[\because (a^2-b^2) = (a+b)(a-b) \right.$
$\left. (a^3-b^3) = (a-b)(a^2+ab+b^2) \right]$

Solution: Given: $\lim_{x \to 1} \frac{x^4-1}{x^3-1}$

$$= \lim_{x \to 1} \frac{(x^2+1)(x^2-1)}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{(x^2+1)(x+1)}{(x^2+x+1)} = \frac{(2)(2)}{3} = \frac{4}{3}$$

$$\therefore \lim_{x \to 1} \frac{x^4-1}{x^3-1} = \frac{4}{3}.$$

H.W

3. $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$: Ans : 3.

$\frac{x^2+x-2}{x^2-x} = \frac{x^2+x-2+x-x}{x(x-1)}$

$= \frac{x^2+2x-2-x}{x(x-1)}$

$= \frac{x(x+2)-1(x+2)}{x(x-1)}$

$= \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} = \frac{3}{1} = 3$

4. If $\lim_{x \to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x \to 1} f(x)$

$\lim_{x \to a} [f(x) g(x)]$
$= \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

Solution: Given: $\lim_{x \to 1} \frac{f(x)-8}{x-1} = 10$

$$\Rightarrow \lim_{x \to 1} (f(x)-8)\left(\frac{1}{x-1}\right) = 10$$

$$\Rightarrow \lim_{x \to 1} (f(x)-8) . \lim_{x \to 1} \left(\frac{1}{x-1}\right) = 10.$$

$$\Rightarrow \lim_{x \to 1} f(x) - 8 = 10 \cdot \lim_{x \to 1} (x-1) = 10(1-1) = 0$$

$$\therefore \lim_{x \to 1} f(x) = 8.$$

5. If $\lim_{x \to 0} \frac{f(x)}{x^2} = 5$, Find the following limits.

a) $\lim_{x \to 0} f(x)$ $= 0$. b) $\lim_{x \to 0} \frac{f(x)}{x}$ $= 0$.

Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$ when x is near 'a'

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

1. use the squeeze theorem, find the value of $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution: Given: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

W.K.T $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$$\Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0$$

$$\Rightarrow \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \quad [\because \text{By S.T}].$$

2. $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$

Solution: **Given:** $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$.

W.K.T, $-1 \le \cos\left(\frac{2}{x}\right) \le 1$.

$$\Rightarrow -x^4 \le x^4 \cos(2/x) \le x^4$$

$$\lim_{x\to 0} (-x^4) = \lim_{x\to 0} x^4 = 0$$

$$\Rightarrow \lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0 \quad [\text{By S.T}].$$

H.W.

3. $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

4. $\lim_{x\to 0} \sqrt{x^3+x^2}\, \sin\left(\frac{\pi}{x}\right)$.

Special Limits:

1. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$ for all rational values of n.

2. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ $\qquad \theta$ is measured in radians.

3. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ $\qquad$ for all rational values of n.

1. **Evaluate:** $\lim_{x\to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2}$ $\qquad$ [U.Q Jan: 2016]

Solution: Given: $\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\pi/2} \frac{2\cos^2 x}{(\pi-2x)^2}$

$$= \lim_{x\to\pi/2} \frac{2\sin^2(\pi/2 - x)}{2^2(\pi/2 - x)^2} = \lim_{x\to\pi/2} \frac{1}{2}\left[\frac{\sin(\pi/2 - x)}{\pi/2 - x}\right]^2$$

$$= \lim_{(x-\pi/2)\to 0} \frac{1}{2}\left[\frac{\sin(x-\pi/2)}{(x-\pi/2)}\right]^2 \qquad \text{Let } \theta = x - \pi/2,$$

$$= \lim_{\theta\to 0} \frac{1}{2}\left(\frac{\sin\theta}{\theta}\right)^2 = \frac{1}{2}\left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right]^2 = \frac{1}{2}(1)^2$$

$$\left[\because \frac{\cos 0}{1} = \frac{1}{1} = 1\right]$$

$$= 1/2.$$

2. H.W. Find $\lim_{x\to 0} (1+x)^{1/x}$. $[\because \text{put } x = 1/n]$.

1.3 Continuity:

Continuous:

A function 'f' is continuous at a number 'a' if $\lim_{x \to a} f(x) = f(a)$

Discontinuous:

A function 'f' is discontinuous at a number 'a' if $\lim_{x \to a} f(x) \neq f(a)$.

Right Limit:

A function 'f' is continuous from the right at 'a' if $\lim_{x \to a^+} f(x) = f(a)$.

Left Limit: A function 'f' is continuous from the left at 'a' if $\lim_{x \to a^-} f(x) = f(a)$.

a) Locate the discontinuity of the function:

1. Explain the function $f(x) = \begin{cases} \cos x, & \text{if } x < 0 \\ 0, & \text{if } x = 0. \\ 1-x^2, & \text{if } x > 0 \end{cases}$,

$a = 0$ is discontinuous at 'a'.

Solution: Given: $f(x) = \begin{cases} \cos x, & x < 0 \\ 0, & x = 0 \\ 1-x^2, & x > 0. \end{cases}$

W.K.T $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \cos x = \cos 0 = 1$$

$$f(0) = 0$$

$$\therefore \lim_{x \to 0} f(x) \neq f(0).$$

$\therefore f(x)$ is discontinuous at 'a'.

2. Locate the discontinuous of the function $y = \frac{1}{1+e^{1/x}}$.

Solution: Given: $f(x) = \frac{1}{1+e^{1/x}}$.

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{1}{1+e^{1/h}} = 0$$

$$f(0) = \frac{1}{1-e^{1/0}} = 0$$

$e^{1/0}$, e^{∞}, $\frac{1}{1+\infty} = 0$, $\frac{1}{\infty} = 0$, $e^{0} = 1$, $e^{-\infty} = 0$, $e^{\infty} = \infty$, $\frac{1}{0} = \infty$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

So $f(x)$ is discontinuous at $x = 0$.

3. Explain the function is continuous at 2.

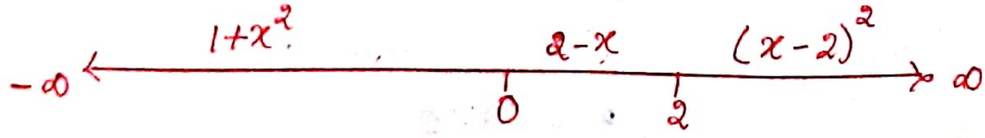
a) $f(x) = \frac{x^3-8}{x^2-4}$ [3] b) $f(x) = \frac{x^2-7x+10}{x-2}$. [-3]

4. Find the domain where the function f is continuous. Also find the numbers at which the function f is discontinuous, where

[U.Q - 2016, 2015]

$$f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2. \end{cases}$$

Solution:



At $x=0$, $(2-x) = 2$

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x^2) = 1 \to ①$$

$$f(0^-) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1 \to ②$$

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2 \to ③$$

From ①, ②, ③ we get,

$$f(0^-) = f(0) \neq f(0^+).$$

So f is continuous on the left at $x=0$

f is discontinuous on the right at $x=0$.

Hence, f is discontinuous at $x=0$.

At $x=2$,

$$f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} (2-x) = 0 \to ①$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 0 \to ②$$

$$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2 = 0 \to ③$$

From ①, ②, ③ we get,

$$f(2^-) = f(2) = f(2^+).$$

Hence f is continuous at $x=2$.

The domain of f is $(-\infty, 0) \cup (0, \infty)$.

5. Find the numbers that at which f is discontinuous, At which of numbers is f Continuous from the right from the left or neither? when

$$f(x) = \begin{cases} x+2, & x<0 \\ e^x, & 0 \le x \le 1 \\ 2-x, & x>1. \end{cases}$$

$\left.\begin{array}{l} f(0)^- = 2 \\ f(0) = 1 \\ f(0)^+ = 1 \end{array}\right\}$ dis at $x=0$, $\left.\begin{array}{l} f(1)^- = e^1 = e \\ f(1) = e^1 = e \\ f(1)^+ = 2-1 = 1 \end{array}\right\}$ dis at $x=1$.

$\therefore f$ is dis. at $x=0$, $x=1$.

$\therefore f$ is cont. at $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

b) function is continuous in a given interval:

1. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Solution:

Given: $f(x) = 1 - \sqrt{1-x^2}$

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) ; \quad \lim_{x \to 1^-} f(x) = 1 = f(1).$$

So, f is continuous from the right at -1

f is continuous from the left at 1

$\therefore f$ is continuous on $[-1, 1]$.

2. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)\,g(x)] = 36$. Find $f(2)$.

Solution:

Given: $\lim_{x \to 2} [3f(x) + f(x)\,g(x)] = 36$, $g(2) = 6$.

$$\Rightarrow 3 \lim_{x \to 2} f(x) + \lim_{x \to 2} f(x)\,g(x) = 36.$$

$$3f(2) + f(2) \cdot g(2) = 36$$

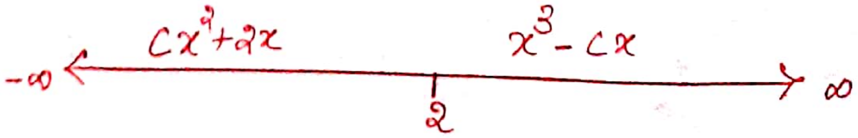
$$\Rightarrow f(2)\,[3+6] = 36 \quad \Rightarrow f(2)\,(9) = 36$$

$$f(2) = 4.$$

3. For what value of the constant 'c' is the function 'f' continuous at $(-\infty, \infty)$. [Jan : 2018]

$$f(x) = \begin{cases} cx^2 + 2x, & x < 2 \\ x^3 - cx, & x \geq 2. \end{cases}$$

Solution:



At $x=2$, Given: f is continuous.

$$\Rightarrow f(2^-) = f(2) = f(2^+) \rightarrow ①$$

$$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left[x^3 - cx\right] = 8-2c \rightarrow ⓐ$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left[cx^2+2x\right] = 4c+4 \rightarrow ⓑ$$

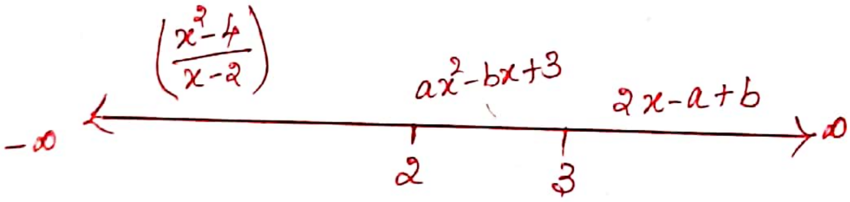
$$① \Rightarrow 8-2c = 4c+4 \quad [\text{by ⓐ \& ⓑ}]$$

$$6c = 4$$

$$c = 2/3.$$

4. If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x<2 \\ ax^2-bx+3, & 2\le x<3 \\ 2x-a+b, & x\ge 3 \end{cases}$ is continuous for all real x, find the values of a and b.

Solution:



At $x=2$, Given: $f(2^-) = f(2) = f(2^+) \rightarrow ①$

$$f(2)^+ = \lim_{x\to 2} f(x) = \lim_{x\to 2}(ax^2-bx+3) = 4a-2b+3 \rightarrow ⓐ$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^-}(x+2) = 4 \rightarrow ⓑ$$

$$① \Rightarrow 4a-2b+3 = 4$$

$$\Rightarrow 4a-2b = 1 \rightarrow ⓒ$$

At $x=3$, Given: $f(3^-) = f(3) = f(3^+) \rightarrow ②$

$$f(3) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(2x-a+b) = 6-a+b \rightarrow ⓓ$$

$$f(3^-) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(ax^2-bx+3) = 9a-3b+3 \rightarrow ⓔ$$

$② \Rightarrow \quad 6-a+b = 9a-3b+3$

$\Rightarrow \quad 10a-4b=3 \rightarrow ⓕ$

Solving ⓒ & ⓕ we get,

$$4a-2b=1$$
$$10a-4b=3$$
$$a=\frac{1}{2}, \; b=\frac{1}{2}.$$

5. Show that f is continuous on $(-\infty,\infty)$.

$$f(x)=\begin{cases} \sin x, & x<\pi/4 \\ \cos x, & x \geq \pi/4 \end{cases}$$

Solution:

At $x=\pi/4$ To prove: $f\left(\frac{\pi}{4}^-\right)=f\left(\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}^+\right) \rightarrow Ⓐ$

$$f(\pi/4)=\lim_{x\to\pi/4} f(x)=\lim_{x\to\pi/4}\cos x=\cos 45°=\frac{1}{\sqrt{2}} \rightarrow ①$$

$$f(\pi/4^-)=\lim_{x\to\pi/4^-} f(x)=\lim_{x\to\pi/4^-}\sin x=\sin 45°=\frac{1}{\sqrt{2}} \rightarrow ②$$

$$f\left(\frac{\pi}{4}^+\right)=\lim_{x\to\frac{\pi}{4}^+} f(x)=\lim_{x\to\pi/4^+}\cos x=\cos 45°=\frac{1}{\sqrt{2}} \rightarrow ③$$

From ① & ② & ③ we get,

$$f(\pi/4^-)=f(\pi/4)=f(\pi/4^+)$$

$\therefore f$ is continuous on $(-\infty,\infty)$.

6. Show that f is continuous on $(-\infty,\infty)$, $f(x)=\begin{cases} x^2, & x<1 \\ \sqrt{x}, & x\geq 1. \end{cases}$

Ans:

To prove: $f(1^-)=f(1)=f(1^+)$

Ans: 1.

$\therefore f$ is continuous on $(-\infty,\infty)$.

1.4 Derivatives:

Tangent Line:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists.

Derivative: The derivative of a function f at a number 'a', denoted by $f'(a)$ is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

NOTE:

The equation of the tangent line is,

$$(y - y_1) = m(x - x_1).$$

$$\text{(or)} \quad y - f(a) = f'(a)(x - a).$$

$f(x) = y = g(x)$
$f'(x) = y' = g'(x).$
$m = f'(x) = y'$

1. Find an equation of the tangent line to the curve at the given point $y = \sqrt{x}$, $(1, 1)$.

Solution: **Given:** $y = f(x) = \sqrt{x}$, $(1, 1)$

$$y' = f'(x) = \frac{1}{2\sqrt{x}}$$

$$\text{Slope } m = f'(x) = (y')_{(1,1)} = \frac{1}{2}.$$

The equation of the tangent line at $(1, 1)$ is,

$$y - y_1 = m(x - x_1).$$

$$y - 1 = \frac{1}{2}(x - 1). \Rightarrow 2y - 2 = x - 1$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(13)

2. Find the slope of the tangent line to the parabola

(2) $y = 4x - x^2$ at $(1,3)$.

Solution: Given: $y = 4x - x^2$

$$f'(x) = 4 - 2x$$

$$m = f'(1) = 4 - 2$$

$$m = 2.$$

3. Find an equation of the tangent line to the

(3) graph of $y = g(x)$ at $x = 5$, if $g(5) = -3$ and $g'(5) = 4$.

Solution: Let $y = g(x)$. Given: $g(5) = -3$, $g'(5) = 4$.

$$(\text{slope})\ m = y' = g'(x) = g'(x)/_{x=5} = 4$$

$$y = g(x) = g(x)/_{x=5} = -3.$$

$x_1 = 5$, $y_1 = -3$ and $m = 4$.

$\therefore$ The equation of the tangent line is,

$$y - y_1 = m(x - x_1)$$

$$y + 3 = 4(x - 5).$$

$$y = 4x - 23.$$

4. If an equation of the tangent line to the curve

(4) $y = f(x)$ at point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

solution: Given: $y = f(x) = 4x - 5$

$$f'(x) = 4$$

$$f(2) = 8 - 5 = 3.$$

$$m = f'(x)/_{x=2} = f'(2) = 4$$

$\therefore f(2) = 3$, $f'(2) = 4$.

5. Find the tangent line to the equation $x^3+y^3=6xy$ at the point $(3,3)$ and at what point the tangent line horizontal in the first Quadrant. [U.Q : 2018 Jan]

(6)

Solution:

Given: $x^3+y^3=6xy$ at $(3,3)$.

$$\left[\begin{aligned} &\Rightarrow 6xy = x^3+y^3 \\ &\Rightarrow y = \frac{x^3+y^3}{6x} = \frac{x^3}{6x}+\frac{y^3}{6x}\end{aligned}\right] \times$$

$$x^3+y^3=6xy \rightarrow (1)$$

$$3x^2+3y^2y' = 6(1.y+xy')$$

$$3x^2+3y^2y' = 6y+6xy'$$

$$3y^2y'-6xy' = 6y-3x^2$$

$$y' = \frac{6y-3x^2}{3y^2-6x} \quad \text{at } 3,3.$$

$$y' = -1 \Rightarrow \frac{dy}{dx} = -1. \Rightarrow m=-1$$

The equation of the tangent line at $(3,3)$ is,

$$y-y_1 = m(x-x_1)$$

$$y-3 = (-1)(x-3)$$

$$y-3 = -x+3$$

$$\Rightarrow x+y = 6.$$

To find the point of the tangent line horizontal in the first quadrant.

$$y' = 0$$

$$\therefore y' = \frac{6y-3x^2}{3y^2-6x} = 0$$

$$\Rightarrow 6y-3x^2=0 \Rightarrow 2y = x^2$$

$$\therefore y = x^2/2.$$

(4).

$$\text{①} \Rightarrow \quad x^3 + y^3 = 6xy$$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$\Rightarrow x^3 + \frac{x^6}{8} = 3x^3 \quad \Rightarrow \frac{x^{\cancel{6}}}{8} = 2\cancel{x^3}$$

$$\Rightarrow x^3 = 16 \quad \Rightarrow x^3 = (2)^4$$

$$\therefore x = 2^{4/3}$$

$x = 2^{4/3}$ put in $y = x^2/2$

$$\Rightarrow y = \frac{(2^{4/3})^2}{2} = 2^{8/3 - 1}$$

$$y = 2^{5/3}.$$

$\therefore$ The points are $x = 2^{4/3}$, $y = 2^{5/3}$.

6. Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Solution: $y = -2x$.

7. Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
Find equations of the tangent lines at the points $(1,1)$ and $(4, 1/2)$.

1.5 Differentiation Rules:

a) Derivatives of polynomials:

1) Equation of tangent line is $y - y_1 = m(x - x_1)$, $m = \frac{dy}{dx}$.

2) Equation of normal line is $y - y_1 = -\frac{1}{m}(x - x_1)$

1. Find an equation of the tangent line and normal line to the curve the given point $y = 3x^2 - x^3$, $(1, 2)$.

6

Solution: Given : $y = 3x^2 - x^3$, $(1,2)$

$$y = 3x^2 - x^3 \Rightarrow y' = 6x - 3x^2$$

$$m = (y')_{1,2} = 6 - 3 = 3.$$

a) Equation of tangent line is,

$$y - y_1 = m(x - x_1)$$

$$(y - 2) = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$\Rightarrow y = 3x - 1$$

b) Equation of normal line is,

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$(y - 2) = -\frac{1}{3}(x - 1)$$

$$3y - 6 = -x + 1$$

$$3y = -x + 7$$

$$y = -\tfrac{1}{3}x + \tfrac{7}{3}.$$

2. H.W $y = \sqrt[4]{x} = (x)^{1/4}$, Ans: $y = \frac{1}{4}x + \frac{3}{4}$, $y = -4x + 5$.

3. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 7$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Solution: The velocity and acceleration are,

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3.$$

16

$$a(t) = \frac{dv}{dt} = 12t - 10$$

$$[a(t)]_{t=2} = 14 \ cm/s^2$$

4. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

a) The velocity and acceleration as function of t,

b) The acceleration after 2 s and

c) the acceleration when the velocity is 0.

Solution: Given: $s = t^3 - 3t$.

a) $$v = \frac{ds}{dt} = 3t^2 - 3$$

$$a = \frac{dv}{dt} = 6t$$

b) If acceleration after 2 seconds, then

$$\left(\frac{dv}{dt}\right)_{t=2} = 12 \ m/s^2$$

c) Find acceleration when the velocity is 0.

$$v = 0 \rightarrow 3t^2 - 3 = 0 \Rightarrow 3t^2 = 3$$

$$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1.$$

$$\therefore \left(\frac{dv}{dt}\right)_{t=1} = 6 \ m/s^2 \quad [\text{reject } t = -1].$$

Derivatives:

5. Find the first and second derivatives of the function $f(x) = 10x^{10} + 5x^5 - x$.

Solution: Given: $f(x) = 10x^{10} + 5x^5 - x$

$$f'(x) = 10[10x^9] + 5[5x^4] - 1$$

$$= 100x^9 + 25x^4 - 1.$$

$$f''(x) = 100\,[9x^8] + 25\,[4x^3] - 0$$
$$= 900x^8 + 100x^3.$$

6. Find y'' if $x^4 + y^4 = 16$.

Solution: Given: $x^4 + y^4 = 16$.

$$\Rightarrow y^4 = -x^4 + 16.$$
$$\Rightarrow 4y^3 y' = -4x^3 + 0$$
$$\Rightarrow y' = -x^3/y^3$$

$$y'' = -\left[\frac{y^3 3x^2 - x^3 3y^2 y'}{(y^3)^2}\right] = -\left[\frac{3x^2y^3 - 3y^2x^3\left(-x^3/y^3\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 + 3x^6/y}{y^6}\right] = -\left[\frac{3x^2y^3}{y^6} + \frac{3x^6}{y^7}\right] = \left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$$

$$= -\left[\frac{3x^2(-x^4+16) + 3x^6}{y^7}\right] = -\left[\frac{-3x^6 + 48x^2 + 3x^6}{y^7}\right]$$

$$y'' = \frac{-48x^2}{y^7}.$$

7. Find a second degree polynomial p such that $p(2) = 5$, $p'(2) = 3$, $p''(2) = 2$.

Solution: Given: $p(2) = 5$, $p'(2) = 3$, $p''(2) = 2$.

Let $p(x) = ax^2 + bx + c \rightarrow (1)$

$p'(x) = 2ax + b \rightarrow (2)$

$p''(x) = 2a \rightarrow (3)$

Given: $p''(2) = 2 \Rightarrow p''(2) = 2a = 2 \Rightarrow a = 1$

$p'(2) = 3 \Rightarrow p'(2) = 4a + b = 3$

$\Rightarrow 4 + b = 3$

$b = -1$

$p(2) = 5 \Rightarrow p(2) = 4a + 2b + c = 5 \Rightarrow 4 - 2 + c = 5 \Rightarrow c = 3.$

$\therefore (1) \Rightarrow p(x) = x^2 - x + 3$

1.6 Maxima and Minima of functions of one variable: ⑤

a) Absolute Maximum and absolute Minimum:

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the

* absolute Maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
* absolute Minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition: The number $f(c)$ is a

* local Maximum value of f if $f(c) \geq f(x)$ when x is near c.
* local Minimum value of f if $f(c) \leq f(x)$ when x is near c.

Extreme value Theorem:

If f is continuous on a closed interval $[a,b]$, then f attains an absolute Maximum value $f(c)$ and an absolute Minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Fermat's Theorem:

If f has a local Maximum or Minimum at c and if $f'(c)$ exists then $f'(c)=0$.

Definition:

A Critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method:

To find the absolute Maximum and Minimum values of a continuous function f on a closed interval $[a, b]$.

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the end points of the interval.
3. The largest of the values from steps 1 and 2 is the absolute Maximum value; the smallest of these values is the absolute Minimum value.

1. Find the critical values of the function:

$f(x) = 5x^2+4x.$

Solution: Given: $f(x) = 5x^2+4x \rightarrow (1)$

Critical numbers of f occur at $f'(x)=0 \rightarrow (2)$

$$f'(x) = 10x+4 \rightarrow (3)$$

$(2) \Rightarrow \quad 10x+4=0 \Rightarrow x = -4/10 \Rightarrow x = -2/5$

$\therefore$ The Critical value $= -2/5$.

2. $g(x) = 2x^3-3x^2-36x.$

Solution: Given: $g(x) = 2x^3-3x^2-36x. \rightarrow (1)$

critical numbers of g occur at $g'(x)=0 \rightarrow (2)$ (16)

$\Rightarrow g'(x) = 6x^2 - 6x - 36$

$(2) \Rightarrow \quad 6x^2 - 6x - 36 = 0$

$\Rightarrow x = -2, x = 3.$

$\therefore$ Critical value $= -2, 3.$

3. $f(x) = x^2 e^{-3x}$. Ans: $x=0, x = 2/3.$

4. $f(x) = x^2 + 2/x$ Ans: $x = 1.$

5. Find the absolute and local Maximum and minimum values of

$f(x) = \frac{1}{x}, x \geq 1.$

Solution: Given: $f(x) = \frac{1}{x}, x \geq 1.$

x	1	2	3	....
$f(x)$	1	$1/2$	$1/3$	....

Maximum

$f(1) = \frac{1}{1} = 1$ is the absolute Minimum of f.

$f(x) = 1/x = 1/1 = 1$

6. $f(x) = \begin{cases} 1-x & \text{if } 0 \leq x < 2 \\ 2x-4 & \text{if } 2 \leq x \leq 3. \end{cases}$ — ? $c=1$ $f(c)=1$ $f(c)=f(x)$.

Solution:

x	0	1	2	3
$f(x)$	1	0	0	2

$f(3) = 6-4 = 2$ is the absolute Maximum.

$= f(0) = 1-x = 1-0 = 1$ is a Min. $c=2$.

7. $f(x) = |x|, -1 < x < 2.$

Ans: $f(0) = 0$ is the A. Minimum.

No, absolute Maximum.

The closed interval Method.

1. Find the absolute Maximum and Minimum values of $f(x) = x^2, [-2,1]$

Solution: Given: $f(x) = x^2, [-2,1] \rightarrow ①$

Critical numbers of f occur at $f'(x)=0 \rightarrow ②$

$$f'(x) = 2x$$

$② \Rightarrow \quad 2x = 0 \quad \Rightarrow x = 0.$

$\therefore$ Critical value $= 0.$

$f(-2) = 4$ is the a. Maximum value of f
$f(0) = 0$ is the a. Minimum value of f.

	end point	Critical point	end point
x	-2	0	1
$f(x)$	4	0	1

2. $f(x) = 3x^4 - 16x^3 + 18x^2, \ -1 \le x \le 4$

Solution: Given: $f(x) = 3x^4 - 16x^3 + 18x^2. \ -1 \le x \le 4 \rightarrow ①$

Critical numbers of f occur at $f'(x) = 0 \rightarrow ②$

$$f'(x) = 12x^3 - 48x^2 + 36x$$

$② \Rightarrow \quad 12x^3 - 48x^2 + 36x = 0$

$\Rightarrow 12x\left[x^2 - 4x + 3\right] = 0$

$\Rightarrow x = 0, 3, 1.$

$\therefore$ Critical value $= 0, 1, 3.$

	end point	Critical point	Critical point	Critical point	End point
x	-1	0	1	3	4
$f(x)$	a.M 37	0	5	-27	32

$f(3) = -27$ is the a. Minimum value of f
$f(-1) = 37$ is the a. Maximum value of f

3. H.W. $f(x) = x^3 - 3x^2 + 1, \ -\frac{1}{2} \le x \le 4.$ [Ans: C.V $= 0, 2$ $f(4) = 17, f(2) = -3$]

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1.6 (b) Increasing / Decreasing test:

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

First Derivative Test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local Maximum at c.

b) If f' changes from negative to positive at c, then f has a local Minimum at c.

c) If f' does not change sign at c (ex: if f' is positive on both sides of c or negative on both sides) then f has no local Maximum or Minimum at c.

Concave upward / concave downward:

If the graph of f lies above all of its tangents on an interval I, then it is called Concave upward on I, If the graph of f lies between all of its tangents on I, it is called Concave downward on I.

Concave upward $\equiv$ Concave downward.

Concavity Test:

a) If $f''(x) > 0$ for all x in I, then the graph of f is Concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is Concave downward on I.

Inflection point:

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Second Derivative Test:

Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local Minimum at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local Maximum at c.

Answer the following questions about the functions whose derivatives are given:

a) What are the critical point of f?

b) On what interval is f increasing or decreasing?

c) At what points, if any, does f assume local Maximum and Minimum values?

d) Find intervals of Concavity and the inflection points.

1. $f(x) = \sin x + \cos x, \; 0 \le x \le 2\pi$ [U.Q Jan 2016]

Solution:

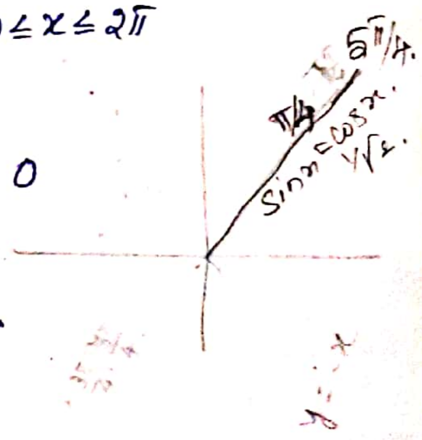
Given: $f(x) = \sin x + \cos x, \; 0 \le x \le 2\pi$

$$f'(x) = \cos x - \sin x$$

$$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \cos x = \sin x.$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}.$$



sin (180+45)

$-\sin 45°$

$\sin(540+\theta) = -\cos\theta$

$\sin[270+\theta] = -\cos\theta$

a) critical points are $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\pi/4 + 360° = 2\pi + \pi/4 = 7\pi/4$

b)

Interval	sign of f'	Behaviour of f
$0 < x < \pi/4$	+	increasing
$\pi/4 < x < 5\pi/4$	−	decreasing
$5\pi/4 < x < 2\pi$	+	increasing.

c) The first derivative test is a local.

(i) Maximum at $\pi/4$, $f(\pi/4) = \sin\pi/4 + \cos\pi/4$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}.$$

(ii) Minimum at $5\pi/4$, $f(5\pi/4) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$

$$= -\sqrt{2}$$

d) $f'(x) = -\sin x - \cos x = -(\sin x + \cos x)$

$f'(x) = 0 \Rightarrow -(\sin x + \cos x) = 0$

$\Rightarrow \sin x = -\cos \boxed{x}$ $(270+\theta)$

$\cos(90+45)$

$= -\sin 45 = \frac{\pi}{2} + \pi/4 = 3\pi/4$ $\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$. $\sin(270+45)$

$= -\cos 45 = \frac{3\pi}{2} + \pi/4 = \frac{7\pi}{4}$

$\pi/4 + \pi$ $\quad \pi - \pi/4$

Interval	sign of f''	Behaviour of f
$0 < x < 3\pi/4$	−	Concave down
$\frac{3\pi}{4} < x < 7\pi/4$	+	Concave up
$7\pi/4 < x < 2\pi$	−	concave down

e) Inflection points are,

$$\left(\frac{3\pi}{4}, 0\right)\left(\frac{7\pi}{4}, 0\right)$$

$\left[\because \text{since, } f\left(\frac{3\pi}{4}\right) = 0,\ f\left(\frac{7\pi}{4}\right) = 0.\right]$

2. $f(x) = x^4 - 2x^2 + 3$ [U.Q Jan 2016]

Solution : Given: $f(x) = x^4 - 2x^2 + 3$

$\Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2-1)$

$= 4x(x-1)(x+1)$

$f'(x) = 0 \Rightarrow 4x(x-1)(x+1) = 0$

$\Rightarrow x = 0, x = 1, x = -1.$

a) Critical points are $x = 0, x = 1, x = -1$.

b).

Interval	Sign of f'	Behavior of f.
$-\infty < x < -1$	$-$	decreasing
$-1 < x < 0$	$+$	increasing
$0 < x < 1$	$-$	decreasing
$1 < x < \infty$	$+$	increasing

$\to$ local min at $x=-1$

local max at $x=0$

local min at $x=1$

c) The first derivative test is a local,

(i) a local Maximum at $x = 0$, $f(0) = 3$.

(ii) a local Minimum at $x = \pm 1$, $f(\pm 1) = 2$.

d) $f''(x) = 12x^2 - 4$

$f''(x) = 0 \Rightarrow 12x^2 - 4 = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow 3x^2 = 1$

$\Rightarrow x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3}$.

Interval	Sign of f''	Behaviour of f
$-\infty < x < -1/\sqrt{3}$	$+$	Concave up
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$-$	Concave down
$\frac{1}{\sqrt{3}} < x < \infty$	$+$	Concave up

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e) Inflection points are $\left(\pm \frac{1}{\sqrt{3}}, \frac{22}{9}\right)$

since, $f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{22}{9}$.

H.W.

3. $f(x) = x + 2\sin x, \; 0 \le x \le 2\pi$

4. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ $\qquad$ $3x^4 - 16x^3 + 18x^2, \; -1 \le x \le 4$.

5. Find the local Maximum and Minimum values of f using both the first and second derivative tests. $y = x^4 - 4x^3$ with respect to concavity and points of inflection.

Solution: Given: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0$

$\Rightarrow 4x^2(x-3) = 0$

$\Rightarrow x = 0, \; x = 3.$ $\qquad$ $-\infty \; (-) \; 0 \; (-) \; 3 \; + \; \infty$ min.

a) $\therefore$ The critical values are $0, 3$.

b)

Interval	Sign of f'	Behaviour of f
$(-\infty, 0)$	$-$	decreasing
$(0, 3)$	$-$	decreasing
$(3, \infty)$	$+$	Increasing

c) First derivative test tells us that f does not have a local Maximum or Minimum at 0.

d) $f''(x) = 12x^2 - 24x$

$f''(x) = 0 \Rightarrow 12x^2 - 24x = 0$

$\Rightarrow 12x(x-2) = 0$

$\Rightarrow x = 0, \; x = 2.$

Interval	$f''(x)$	Behaviour of f
$(-\infty, 0)$	+	Concave up
$(0, 2)$	−	Concave down
$(2, \infty)$	+	Concave up

e) Inflection points are,

$$(0,0) \quad (2,-16)$$

$\because$ Since, $f(0) = 0$, $f(2) = -16$.

f) The second derivative test:

$$f''(c) = 0.$$

$$f''(x) = 12x^2 - 24x.$$

$f'(3) = 0$, $f''(3) > 0$, $f(3) = -27$ is a local minimum.

The second derivative test gives no information about the critical number 0.

Since $f'(0) = 0$, $f''(0) = 0$.

But first derivative test gives f does not have a local Maximum or minimum at 0.

H.W

6. Find the local Maximum and Minimum values of $f(x) = x^5 - 5x + 3$ using both the first and second derivative tests.

Ans: $f'(x) = 5x^4 - 5$ $\quad f'(1) = 0$, $f'(-1) = 0$. $f''(1) = 20$, $f''(-1) = -20$.

$x = 1, -1.$

1st derivative:

i) Max $f(-1) = 7$

ii) Min $f(1) = -1$

2nd derivative:

i) Max $f(-1) = 7$

ii) Min $f(1) = -1.$

① For the function $f(x) = 2x^3 + 3x^2 - 36x$

(i) Find the intervals on which it is increasing or decreasing.

(ii) Find the local max & local min. values of f.

(iii) Find the intervals of concavity & the inflection points.

$\longrightarrow$ (i) Given $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$f'(x) = 0 \Rightarrow 6(x^2 + x - 6) = 0 \Rightarrow 6(x+3)(x-2) = 0.$

$\Rightarrow x = -3, 2.$

$+ \qquad - \qquad +$

$-\infty \qquad -3 \qquad 2 \qquad \infty$

$\downarrow \qquad \downarrow \qquad \downarrow$

Take $x = -4$	$x = 0$	$x = 3$
$f'(x) = 6(-4+3)(-4-2)$ $= 6(-1)(-6)$ $= 36$	$f'(x) = 6(3)(-2)$ $= -36$	$f'(x) = 6(6)(1)$ $= 36$

Interval	Nature of $f'(x)$	f
$(-\infty, -3)$	+	$\uparrow$
$(-3, 2)$	−	$\downarrow$
$(2, \infty)$	+	$\uparrow$

$\Rightarrow f$ is max. at $x = -3$

$\Rightarrow f$ is min at $x = 2$.

(ii) $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = -54 + 27 + 108 = 81$

$f(2) = 2(2)^3 + 3(2)^2 - 36(2) = 16 + 12 - 72 = -44$

(iii) $f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6 \cancel{=0}$

$f''(x) = 0 \Rightarrow 12x + 6 = 0$

$\Rightarrow x = \frac{-6}{12} = \frac{-1}{2}$.

At $x = -\frac{1}{2}$,

$f(x) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 36\left(-\frac{1}{2}\right)$

$= -\frac{37}{2}$.

$-$ $+$

$-\infty$ $-\frac{1}{2}$ ∞

Take

$\downarrow$	$\downarrow$
$x = -1$	$x = 0$
$f''(x) = 12(-1) + 6 = -6$	$f''(x) = 6$

Interval	Nature of f''	f
$\left(-\infty, -\frac{1}{2}\right)$	$-$	Concave downward
$\left(-\frac{1}{2}, \infty\right)$	$+$	Concave upward

Here inflection point is $\left(-\frac{1}{2}, -\frac{37}{2}\right)$.

(2) $x^4 - 4x^3$

(3) $3x^4 - 4x^3 - 12x^2 - 5$

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(i) $\longrightarrow$ $f(x) = 3x^4 - 4x^3 - 12x^2 - 5.$

$f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0.$

$\Rightarrow 12x(x^2 - x - 2) = 0.$

$\Rightarrow 12x(x-2)(x+1) = 0$ ✓

$\Rightarrow x = 0, 2, -1.$

$\Rightarrow x = -1, 0, 2$ (critical Points)

	$-$	$+$	$-$	$+$
$-\infty$ (	-1	0	2	∞)

Take $x = -2$

$f'(x) = 12(-2)(-2-2)(-2+1)$

$= 12(-2)(-4)(-1)$

$= -96$

$x = -\frac{1}{2}$

$f' = \left[12(-\frac{1}{2})(-\frac{1}{2}-2) \cdot (-\frac{1}{2}+1)\right]$

$= 12(-\frac{1}{2})(-\frac{5}{2})(\frac{1}{2})$

$= +\frac{15}{2}$

$x = 1$

$f' = 12(1)(1-2)(1+1)$

$= 12(1)(-1)(2)$

$= -24$

$x = 3$

$f' = 12(3)(3-2)(3+1)$

$= 12(3)(+1)(4)$

$= +144$

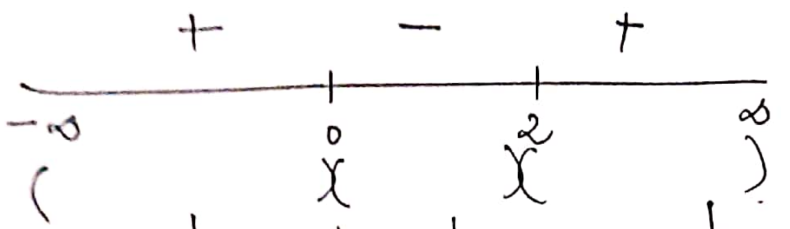
Interval	f'	f
$(-\infty, -1)$	$-$	Decreasing
$(-1, 0)$	$+$	Increasing
$(0, 2)$	$-$	Decreasing
$(2, \infty)$	$+$	Increasing.

$$f''(x) = 0 \Rightarrow 12x^2 - 24x = 0.$$

$$\Rightarrow 12x(x-2) = 0 \checkmark$$

$$\Rightarrow x = 0, 2$$

$$\left[\begin{array}{l} 12x(x-2) = 0. \\ \Rightarrow 12 \neq 0,\ x = 0,\ x-2 = 0 \end{array}\right.$$



Take

$x = -1$	$x = 1$	$x = 3$
$f''(x) = 12(-1)(-1-2)$	$f'' = 12(1)(1-2)$	$f'' = 12(3)(3-2)$
$= 12(-1)(-3)$	$= 12(1)(-1)$	$= 12(3)(1)$
$= +36$	$= -12$	$= +36$

Interval	f''	f
$(-\infty, 0)$	$+$	Concave upwards
$(0, 2)$	$-$	Concave downwards
$(2, \infty)$	$+$	Concave upwards

$\boxed{\text{Try}}$ (3) $f(x) = 3x^4 - 4x^3 - 12x^2 - 5$

$\rightarrow$ or move to next Page.

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(ii) $f'(x) = 12x^3 - 12x^2 - 24x$

$f''(x) = 36x^2 - 24x - 24$

$f''(x) = 0 \Rightarrow 36x^2 - 24x - 24 = 0.$

$\Rightarrow 12(3x^2 - 2x - 2) = 0.$ ✓

-6

-2

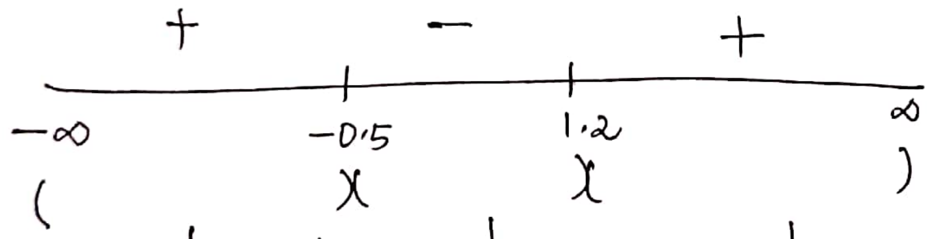
$3x^2 - 2x - 2 = 0.$

$\Rightarrow \quad x = \dfrac{2 \pm \sqrt{4+24}}{6}$

$= \dfrac{2 \pm \sqrt{28}}{6} = \dfrac{2 \pm \sqrt{4 \times 7}}{6} = \dfrac{2 \pm 2\sqrt{7}}{6} = \dfrac{1 \pm \sqrt{7}}{3}.$

$\simeq \dfrac{1 \pm 2.6}{3} \simeq \dfrac{3.6}{3}, \dfrac{-1.6}{3}$

$\simeq 1.2, -0.5$



Take

$x = -1$	$x = 0$	$x = 2$
$f'' = 12(3+2-2)$	$f'' = 12(-2)$	$f'' = 12(12-4-2)$
$= +36$	$= -24$	$= 12(6)$
		$= +72$

Interval	f''	f
$(-\infty, -0.5)$	$+$	Increasing
$(-0.5, 1.2)$	$-$	Decreasing
$(1.2, \infty)$	$+$	Increasing

Differentiation Rules:

1. Derivative of a constant function is zero.

(ie) $\frac{d}{dx}(c) = 0.$

2. The Power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

3. The constant multiple rule

$$\frac{d}{dx}[c\, f(x)] = c\frac{d}{dx}f(x)$$

4. The Sum rule

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

5. The difference rule

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

6. Derivative of the natural exponential function

$$\frac{d}{dx}(e^x) = e^x.$$

7. The product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\cdot\frac{d}{dx}[g(x)] + g(x)\cdot\frac{d}{dx}[f(x)]$$

(or)

$$\frac{d}{dx}(uv) = uv' + vu'$$

8. The quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}[f(x)] - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(or)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Derivatives of Polynomials:

1. Differentiate the following function

(i) $f(x) = 7$ (ii) $f(x) = x^2$ (iii) $y = t^4$

(iv) $y = \sqrt{x}$ (v) $y = ax^2 + bx + c$

(vi) $y = x^2(1-2x)$. (vii) $y = 1/x^2$

Soln:

(i) Given: $f(x) = 7$

$$f'(x) = 0$$

(ii) Given: $f(x) = x^2$

$$f'(x) = 2x^{2-1}$$
$$= 2x$$

(iii) Given: $y = t^4$

$$\frac{dy}{dt} = y' = 4t^{4-1} = 4t^3$$

(iv) Given: $y = \sqrt{x} = x^{1/2}$

$$\frac{dy}{dx} = y' = \frac{1}{2} x^{1/2 - 1}$$

$$= \frac{1}{2} x^{\frac{1-2}{2}}$$

$$= \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(v) Given: $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

(vi) Given: $y = x^2(1-2x) = x^2 - 2x^3$

$$\frac{dy}{dx} = 2x - 2(3)x^2$$

$$= 2x - 6x^2$$

(vii) Given: $y = 1/x^2 = x^{-2}$

$$\frac{dy}{dx} = -2\, x^{-2-1}$$

$$= -2x^{-3}$$

$$= -2/x^3$$

2. Find the first and second derivatives of the following function of $f(x) = 10x^{10} + 5x^5 - x$.

Soln: Given: $f(x) = 10x^{10} + 5x^5 - x$

$$f'(x) = 10(10x^9) + 5(5x^4) - 1$$

$$f'(x) = 100x^9 + 25x^4 - 1$$

$$f''(x) = 100(9x^8) + 25(4x^3)$$

$$f''(x) = 900x^8 + 100x^3$$

o

Derivatives of Exponential function:

3. Differentiate the following functions:

(i) $y = e^{2x} - x$ (ii) $e^x + \frac{2}{\sqrt{x}}$

(iii) $y = a^x$ (iv) $y = \frac{xe^x - 1}{x}$

Soln (i) Given: $y = e^{2x} - x$

$$\frac{dy}{dx} = 2e^{2x} - 1.$$

(ii) Given: Let $y = e^x + \frac{2}{\sqrt{x}} = e^x + 2x^{-1/2}$

$$\frac{dy}{dx} = e^x + \cancel{2}(-\tfrac{1}{\cancel{2}})\, x^{-1/2 - 1}$$

$$= e^x - x^{-3/2}.$$

(iii) $y = a^x$

$$y = e^{\log a^x} = e^{x \log a} = e^{(\log a)x}$$

$$y' = \frac{dy}{dx} = e^{(\log a)x} \cdot (\log a) = a^x \log a$$

(iv) Given: $y = \frac{xe^x - 1}{x} = \frac{xe^x}{x} - \frac{1}{x}$

$$y = e^x - 1/x = e^x - x^{-1}$$

$$y' = e^x - (-1)x^{-1-1}$$

$$= e^x + x^{-2} = e^x + 1/x^2$$

Product and Quotient rules:

4. Differentiate the following function $y = xe^{2x}$.

Soln: Given: $y = xe^{2x}$

$$\frac{dy}{dx} = x\frac{d}{dx}(e^{2x}) + e^{2x} \cdot \frac{d}{dx}(x)$$

$$= x(2e^{2x}) + e^{2x}(1)$$

$$= 2xe^{2x} + e^{2x}$$

$$= e^{2x}(2x+1).$$

---

5. If $f(x) = \frac{x^2}{1+2x}$, then find $f'(x)$.

Soln. Given: $f(x) = \frac{x^2}{1+2x}$

$$f'(x) = \frac{(1+2x)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(1+2x)}{(1+2x)^2}$$

$$= \frac{(1+2x)(2x) - x^2(2)}{(1+2x)^2}$$

$$= \frac{2x(1+2x) - 2x^2}{(1+2x)^2}$$

$$= \frac{2x + 4x^2 - 2x^2}{(1+2x)^2}$$

$$= \frac{2x^2+2x}{1+4x^2+4x}$$

$$\boxed{f'(x) = \frac{2x(x+1)}{1+4x^2+4x}}$$

//..

6. Find $f'(x)$ and $f''(x)$ of $f(x) = x^4 e^x$.

Soln: Given $f(x) = x^4 e^x$

$$f'(x) = x^4 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^4)$$

$$= x^4 e^x + e^x(4x^3)$$

$$\therefore f'(x) = x^4 e^x + 4x^3 e^x.$$

$$f''(x) = x^4 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^4) + 4\left[x^3 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^3)\right]$$

$$= x^4 e^x + e^x(4x^3) + 4\left[x^3 e^x + e^x(3x^2)\right]$$

$$= x^4 e^x + 4x^3 e^x + 4x^3 e^x + 12x^2 e^x$$

$$= x^4 e^x + 8x^3 e^x + 12x^2 e^x$$

$$\therefore f''(x) = e^x\left[x^4 + 8x^3 + 12x^2\right].$$

o

7. Find $f'(x)$ of $f(x) = \frac{x^3}{1-x^2}$.

Soln: Given: $f(x) = \frac{x^3}{1-x^2}$

$$f'(x) = \frac{(1-x^2)(3x^2) - x^3(0-2x)}{(1-x^2)^2}$$

$$= \frac{(1-x^2)3x^2 + 2x^4}{(1-x^2)^2} = \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2}$$

$$\therefore f'(x) = \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2}.$$

Derivatives of Trigonometric function:

1. Find the derivatives of the following functions

(i) $y = x^4 - \sin x$ (ii) $y = x^3 \sin x$

(iii) $y = \dfrac{\cos x}{1-\sin x}$ (iv) $y = \dfrac{x \sin x}{1+x}$

Soln:

(i) Given $y = x^4 - \sin x$

$$y' = \frac{dy}{dx} = 4x^3 - \cos x.$$

$$y' = 4x^3 - \cos x.$$

(ii) $y = x^3 \sin x$

$$y' = dy/dx = x^3(\cos x) + \sin x\,(3x^2)$$

$$\therefore y' = x^3 \cos x + 3x^2 \sin x.$$

(iii) Given: $y = \dfrac{\cos x}{1-\sin x}$

$$y' = \frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x\,(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad [\because \sin^2 x + \cos^2 x = 1]$$

$$= \frac{1-\sin x}{(1-\sin x)^{\cancel{2}}} = \frac{1}{1-\sin x}$$

(iv) $y = \dfrac{x\sin x}{1+x}$

$$y' = \frac{dy}{dx} = \frac{(1+x)\left[x(\cos x)+\sin x(1)\right] - x\sin x\,(0+1)}{(1+x)^2}$$

$$= \frac{(1+x)\left[x\cos x+\sin x\right] - x\sin x}{(1+x)^2}$$

$$= \frac{x\cos x+\sin x+x^2\cos x+\cancel{x\sin x}-\cancel{x\sin x}}{(1+x)^2}$$

$$\therefore\ y' = \frac{x\cos x+x^2\cos x+\sin x}{(1+x)^2}.$$

Chain Rule:- $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}}$

Power Rule Combined with Chain Rule:-

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.}$$

Alternatively, $\boxed{\dfrac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1}\cdot g'(x)}$

1. Find the derivatives of the following :

(i) $y = (x^3-1)^{100}$ (ii) $y = \frac{1}{(t^4+1)^3}$

Soln: (i) Given: $y = (x^3-1)^{100}$

$$\frac{dy}{dx} = 100\,(x^3-1)^{99}\,\frac{d}{dx}(x^3-1)$$

$$= 100\,(x^3-1)^{99}\,(3x^2)$$

$$\frac{dy}{dx} = 300\,x^2\,(x^3-1)^{99}.$$

(ii) Given: $y = \frac{1}{(t^4+1)^3} = (t^4+1)^{-3}$

$$\frac{dy}{dx} = -3\,(t^4+1)^{-4}\,\frac{d}{dt}(t^4+1)$$

$$= -3\,(t^4+1)^{-4}\,(4t^3)$$

$$\frac{dy}{dx} = -12\,t^3\,(t^4+1)^{-4}$$

2. $y = \cos(x^2)$

$$\frac{dy}{dx} = -\sin(x^2)\,\frac{d}{dx}(x^2)$$

$$= -\sin x^2\,(2x)$$

$$y' = -2x\sin^2 x$$

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

Limit of a function

A function $f(x)$ tends to a definite limit l as x tends to 'a', if the difference between $f(x)$ and l can be made as small as we like by making x approach sufficiently near 'a' and we write

$$\lim_{x \to a} f(x) = l$$

Left-hand limit of f(x)

The left-hand limit of $f(x)$ as x approaches a is equal to L.

(ie) $\lim_{x \to a^-} f(x) = L$.

Here $x \to a^-$ means $x < a$.

Right-hand limit of f(x)

The Right-hand limit of $f(x)$ as x approaches a is equal to L.

(ie) $\lim_{x \to a^+} f(x) = L$

Here $x \to a^+$ means $x > a$.

Infinite Limits

1. $\lim_{x \to a^-} f(x) = \infty$
2. $\lim_{x \to a^+} f(x) = \infty$
3. $\lim_{x \to a^-} f(x) = -\infty$
4. $\lim_{x \to a^+} f(x) = -\infty$.

Note:-

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

1. $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$

Soln: Given $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$

$$= \frac{2-1}{(1-1)^2} = \frac{1}{0}$$

$$\therefore \lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$$

2. $\lim_{x\to -3^+} \left(\frac{x+2}{x+3}\right)$

Soln: $x \to -3^+ \Rightarrow x$ is close to -3 but larger then -3.

Nr $= x+2$ becomes negative

Dr $= x+3$ becomes positive

Let $x = -2.9$

Nr $= (-2.9+2 = -0.9 = -ve)$

Dr $= (-2.9+3 = 0.1 = +ve)$

$$\therefore \lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty.$$

3. Find the Value of $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Soln: $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$$= \lim_{x\to 1} (x+1) = (1+1)$$

$$= 2$$

4. Find $\lim_{x\to 1} \frac{x^4-1}{x^3-1}$

Soln. $\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{(x^2+1)(x^2-1)}{(x-1)(x^2+x+1)}$

$\left[\because a^2-b^2 = (a+b)(a-b) \quad a^3-b^3 = (a-b)(a^2+ab+b^2)\right]$

$$= \lim_{x\to 1} \frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x\to 1} \frac{(x^2+1)(x+1)}{(x^2+x+1)}$$

$$\therefore \lim_{x\to 1} \frac{x^4-1}{x^3-1} = \frac{(2)(2)}{3} = 4/3$$

---

5. If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$.

Soln. Given: $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$

$$\Rightarrow \lim_{x\to 1} (f(x)-8)\left(\frac{1}{x-1}\right) = 10$$

$$\Rightarrow \lim_{x\to 1} (f(x)-8) \cdot \lim_{x\to 1} \left(\frac{1}{x-1}\right) = 10$$

$$\Rightarrow \lim_{x\to 1} f(x)-8 = 10 \lim_{x\to 1} (x-1)$$

$$= 10(1-1)$$

$$= 0$$

$$\Rightarrow \lim_{x \to 1} f(x) - 8 = 0$$

$$\therefore \boxed{\lim_{x \to 1} f(x) = 8.}$$

H.W 1. Find $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$ Ans: 3.

Special limits

↳ $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$ for all rational value of n.

↳ $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ is measured in radians)

↳ $\lim_{n \to \infty} (1+1/n)^n = e$ for all rational value of n.

6. Evaluate $\lim_{x \to \pi/2} \frac{1+\cos 2x}{(\pi - 2x)^2}$.

Soln:

$$\lim_{x \to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x \to \pi/2} \frac{2\cos^2 x}{(\pi-2x)^2}$$

$$= \lim_{x \to \pi/2} \frac{2\sin^2(\pi/2 - x)}{2^2(\pi/2 - x)^2}$$

$$= \lim_{x \to \pi/2} \frac{1}{2} \left[\frac{\sin(\pi/2 - x)}{\pi/2 - x} \right]^2$$

$$= \lim_{(x - \pi/2) \to 0} \frac{1}{2} \left[\frac{\sin(x - \pi/2)}{(x - \pi/2)} \right]$$

Let $\theta = x - \pi/2$, we get

$$= \lim_{\theta \to 0} \frac{1}{2} \left(\frac{\sin\theta}{\theta} \right)^2 = \frac{1}{2} \left[\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right]^2$$

$$= \frac{1}{2}(1)^2 = \frac{1}{2}$$

$$= \frac{1}{2}$$

---

7. Find $\lim_{x \to 0} (1+x)^{1/x}$

Soln: Given $\lim_{x \to 0} (1+x)^{1/x}$

Put $x = 1/n$, we get

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{n \to \infty} (1 + 1/n)^n = e.$$

Continuity:

A function f is continuous at 'a' if

$$\lim_{x \to a} f(x) = f(a).$$

Note:- If f is continuous at 'a', then

- $f(a)$ is should exist (that is, a is in the domain of f)
- $\lim_{x \to a} f(x)$ exists both on the left and right.
- $\lim_{x \to a} f(x) = f(a)$.

---

1. Show that $f(x) = 3x^2+2x-1$ is continuous at $x=2$

Soln. Given: $f(x) = 3x^2+2x-1$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \left[3(2-h)^2+2(2-h)-1\right]$$

$$= 3(2)^2+2(2)-1$$

$$= 3(4)+4-1$$

$$= 12+4-1$$

$$= 16-1$$

$$\lim_{x \to 2^-} f(x) = 15$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}\left[3(2+h)^2+2(2+h)-1\right]$$

$$= 3(2)^2+2(2)-1$$

$$= 12+4-1$$

$$\lim_{x\to 2^+} f(x) = 15$$

$$\therefore \lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x).$$

Hence, $f(x)$ is continuous at $x=2$.

---

2. Test the continuity of $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1}, & x\neq 1 \\ 1, & x=1 \end{cases}$ at $x=1$.

Soln:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-x}{x^2-1}$$

$$= \lim_{x\to 1}\frac{x\,\cancel{(x-1)}}{(x+1)\,\cancel{(x-1)}}$$

$$= \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{1+1}$$

$$= 1/2$$

$$\therefore f(1) = 1$$

$$\therefore \lim_{x\to 1} f(x) \neq f(1)$$

$\therefore f$ is discontinuous at $x=1$.

---

3. Test the continuity of $f(x) = \begin{cases} e^x, & x < 0 \\ x^2, & x \geq 0. \end{cases}$

Soln

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$

$$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \text{ and}$$

$$\therefore \lim_{x \to 0} f(x) \text{ does not exist}$$

---

4. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

Soln Given: $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$, $g(2) = 6$

$$\Rightarrow 3 \lim_{x \to 2} f(x) + \lim_{x \to 2} f(x)\, g(x) = 36$$

$$\Rightarrow 3f(2) + f(2)g(2) = 36$$

$$\Rightarrow f(2)[3+6] = 36$$

$$\Rightarrow 9 f(2) = 36$$

$$\Rightarrow f(2) = \frac{36}{9}$$

$$\therefore \boxed{f(2) = 4}$$

Q For what value of the constant b is the function f continuous on $(-\infty, \infty)$. $f(x) = \begin{cases} bx^2+2x, & x<2 \\ x^3-bx, & x\geq 2 \end{cases}$.

Soln. The given function $f(x)$ is continuous on $(-\infty,2)$ & $(2,\infty)$.

At $x=2$,

Now $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} bx^2+2x$

$= b(2)^2+2(2)$

$= 4b+4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3-bx$

$= (2)^3-b(2)$

$= 8-2b$

$f(x)$ is continuous at $x=2$.

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\Rightarrow 4b+4 = 8-2b$

$\Rightarrow 4b+2b = 8-4$

$\Rightarrow 6b = 4$

$\Rightarrow b = 4/6$

$\therefore \boxed{b = 2/3}$

6. For what values of the constant 'C' is the function f continuous on $(-\infty, \infty)$ $f(x) = \begin{cases} cx^2+2x, & x<2 \\ x^2-cx, & x \geq 2 \end{cases}$ at $x=2$.

Soln: The given function $f(x)$ is continuous on $(-\infty, 2)$ & $(2, \infty)$.

Now $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} cx^2+2x$

<u>At $x=2$,</u> $= c(2)^2+2(2)$

$= 4c+4$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} x^2-cx$

$= (2)^2 - c(2)$

$= 4-2c$

$f(x)$ is continuous at $x=2$.

$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$\Rightarrow 4c+4 = 4-2c$

$\Rightarrow 4c+2c = 4-4$

$\Rightarrow 6c = 0$

$\therefore \boxed{c=0}$

<u>H.w</u> For what value of the constant 'c' is the function f continuous on $(-\infty, \infty)$

$f(x) = \begin{cases} cx^2+2x, & x<2 \\ x^3-cx, & x\geq 2. \end{cases}$ <u>Ans:</u> $c = 2/3$.

7\. If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2-bx+3, & 2 \le x < 3 \\ 2x-a+b, & x \geq 3 \end{cases}$ is continuous for all x. Find the value of 'a' and 'b'.

Soln The given function $f(x)$ is continuous for all real 'x'.

$\therefore f(x)$ is continuous at $x=2$, $x=3$.

At $x=2$,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$= \lim_{x \to 2^-} (x+2)$$

$$= 2+2 = 4 \longrightarrow (1)$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2-bx+3$$

$$= a(2)^2-b(2)+3$$

$$= 4a-2b+3 \longrightarrow (2)$$

At $x=3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2-bx+3$$

$$= a(3)^2-b(3)+3$$

$$= 9a-3b+3 \longrightarrow (3)$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b$$

$$= 2(3) - a + b$$

$$= 6 - a + b \longrightarrow (4)$$

From (1) and (2), $4a - 2b + 3 = 4$

$$4a - 2b = 4 - 3$$

$$4a - 2b = 1 \longrightarrow (5)$$

From (3) and (4), $6 - a + b = 9a - 3b + 3$

$$10a - 4b = 6 - 3$$

$$10a - 4b = 3 \longrightarrow (6)$$

Solving (5) and (6), we get

$$10a - 4b = 3 \longrightarrow (7)$$

$$4a - 2b = 1 \longrightarrow (8)$$

$(7) \Rightarrow \quad 10a - \cancel{4b} = 3 \longrightarrow (9)$

$(8) \times 2 \Rightarrow \quad 8a - \cancel{4b} = 2 \longrightarrow (10)$

$\quad (-) \quad (+) \quad (-)$

$(9) - (10) \Rightarrow \quad 2a = 1$

$$\boxed{a = 1/2}$$ in (8), we get

$\therefore (8) \Rightarrow \quad \cancel{4}^{2}(1/\cancel{2}) - 2b = 1$

$$2 - 2b = 1$$

$$-2b = 1 - 2 \Rightarrow -2b = -1$$

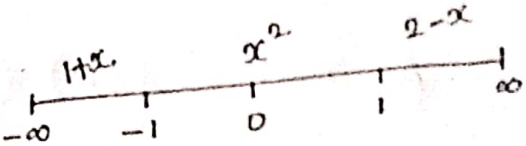
$$\therefore \boxed{b = 1/2}$$

8. Sketch the graph of the function

$$f(x)=\begin{cases} 1+x, & x<-1 \\ x^2, & -1\le x\le 1 \\ 2-x, & x\geq 1. \end{cases}$$

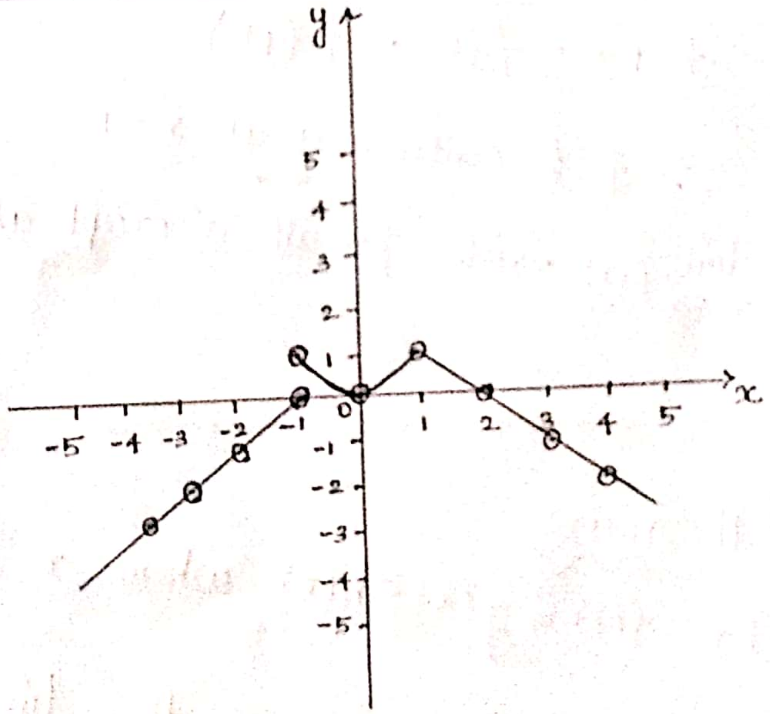
and use it to determine the value of 'a' for which $\lim_{x\to a} f(x)$ exists.

Soln:



$1+x;\ x<-1$ $\qquad$ $x^2;\ -1\le x\le 1$ $\qquad$ $2-x^2;\ x\geq 1$

x :	-2	-3	-4	-1	0	1	1	2	3	4
$f(x)$:	-1	-2	-3	1	0	1	1	0	-1	-2



$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (1+x) \qquad \Rightarrow f(-1^-)$$

At $x=-1$

$$= 1-1$$

$$= 0$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} x^2 \qquad \Rightarrow f(-1)$$

$$= (-1)^2 = 1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-1)^2 = 1. \Rightarrow f(-1^+)$$

Here $f(-1^-) \neq f(-1) = f(-1^+)$

$\therefore$ f is discontinuous at $x=-1$.

<u>At $x=1$</u>

$$f(1^-) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = (1)^2 = 1$$

$$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = (1)^2 = 1$$

$$f(1^+) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1.$$

Here, $f(1^-) = f(1) = f(1^+)$

$\therefore$ f is continuous at $x=1$.

Hence, $\lim_{x\to a} f(x)$ exists for all 'a' except at $a=-1$.

o

Squeeze theorem:

If $f(x) \leq g(x) \leq h(x)$ where x is near 'a'

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

o

Discontinuous:

A function f is discontinuous at a number 'a' if $\lim_{x\to a} f(x) \neq f(a)$.

o

PRIYADARSHINI ENGINEERING COLLEGE

(Listed in 2(f) & 12(B) Sections of UGC,

Approved by AICTE, New Delhi, and Affiliated to Anna University, Chennai)

MA8151-ENGINEERING MATHEMATICS-I

UNIT-I DIFFERENTIAL CALCULUS

MULTIPLE CHOICE QUESTIONS

1.The Domain of the function f(x) = $\sqrt{x+3}$

A) $(-\infty,\infty)$ B) $[-3,\infty)$ ✓ C) $[-2,\infty)$ D) $(-3,2)$

2. The Domain of the function f(x) = $\sqrt{4-x}-\sqrt{3+x}$

A) $[-2,4]$ B) $(-3,4]$ C) $[-2,4)$ D) $[-3,4]$ ✓

3.The Domain of the function f(x) = $\dfrac{2x^2-5}{x^2+x-6}$

A) $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ ✓ B) $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$

C$(-\infty,-2)\cup(-2,\infty)$ D) None of the above

4.The function f(x) = $\dfrac{x}{x^2+1}$ **is**

A)Odd function ✓ B)Even function

C) neither odd nor Even D) Not define

5. The function f(x) = $x|x|$ **is**

A)Odd function ✓ B)Even function

C) neither odd nor Even D) Not define

6.The function f(x) = $\dfrac{x}{x+1}$ **is**

A)Odd function B)Even function

C) Neither odd nor Even ✓ D) Not define

7.Which of the following statement is true?

A) A function f(x) is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1<x_2$ in I ✓

B) A function f(x) is called Decreasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1<x_2$ in I

C) A function f(x) is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1>x_2$ in I

D) A function f(x) is called Decreasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1>x_2$ in I

8.The value of $\lim_{x\to 2}(x^2 - x + 2)$

A) 0 B) 6 C) ∞ D) 4 ✓

9.The value of $\lim_{x\to 1}\frac{x^2-1}{x-1}$

A) 0 B) 6 C) ∞ D) 2 ✓

10.The value of $\lim_{x\to 0}\frac{\sqrt{x^2+9}\ -3}{x^2}$

A) 0 B) $\frac{1}{8}$ C) $\frac{1}{2}$ D) $\frac{1}{6}$ ✓

11.The value of $\lim_{x\to 0}\frac{9^x-5^x}{x}$

A) $\log\frac{9}{5}$ ✓ B) $\log\frac{5}{9}$ C) $\log\frac{1}{2}$ D) ∞

12.The value of $\lim_{x\to 0}\frac{|x|}{x}$

A) 0 B) 1 C) $\log\frac{1}{2}$ D) Does not exist ✓

13. The value of $\lim_{x\to 0}\frac{|\sin x|}{\sin x}$

A) 0 B) 1 ✓ C) $\log\frac{1}{2}$ D) Does not exist

14. The value of $\lim_{x\to\infty}\frac{x^2+x}{x-3}$

A)-2 B) 6 C) ∞ D) $-\infty$ ✓

15. The value of $\lim_{h\to 0}\frac{(3+h)^2-9}{h}$

A)-2 B) 6 ✓ C) 4 D) $-\infty$

16.Which of the following statement is true According to Sequeeze theorem?

A) If f(x)$\le g(x)\le h(x)$ when x is near a and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)=L$ ✓

B) If f(x)$\le h(x)\le g(x)$ when x is near a and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)=L$

C) If g(x)$\le f(x)\le h(x)$ when x is near a and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)=L$

D) If h(x)$\le f(x)\le g(x)$ when x is near a and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)=L$

17.Discuss the continuity of the function

$$f(x)=\begin{cases}\frac{x^2-x-2}{x-2} & if \quad x\neq 2\\ 1 & if \quad x=2\end{cases}$$

A) The function f is continuious at x =2

B) The function f is discontinuious at x =2 ✓

C) The function f is continuious Everywhere.

D) The function f is discontinuious Everywhere.

18.The value of constant 'c' is the function continuous on$(-\infty,\infty)$

f(x)= $\begin{cases} cx^2+2x & if \quad x<2\\ x^3-cx & if \quad x\ge 2\end{cases}$

A) $\frac{3}{5}$ B) $\frac{5}{3}$ C) $\frac{2}{3}$ ✓ D) $\frac{4}{3}$

19.The value of a and b that makes the function continuous everywhere

f(x)= $\begin{cases} \dfrac{x^2-4}{x-2} & if \quad x<2 \\ ax^2-bx+3 & if \quad 2\le x<3 \\ 2x-a+b & if \quad x\ge 3 \end{cases}$

A) a = $\frac{3}{2}$,b= $\frac{1}{2}$ B) a = $\frac{2}{3}$,b= $\frac{1}{2}$

C) a = $\frac{1}{2}$,b= $\frac{1}{2}$ ✓ D) a = $\frac{3}{2}$,b= $\frac{3}{2}$

20.The value of constant 'c' is the function continuous on($-\infty,\infty$)

f(x)= $\begin{cases} \dfrac{x^2-x-2}{x-2} & if \quad x\ne 2 \\ c & if \quad x=2 \end{cases}$

A) 3 ✓ B) 4 C) 5 D) 6

21.If f(x) = $\sqrt{x}$**, then the derivative of f(x) is**

A) $\frac{1}{2}\sqrt{x}$ B) $\frac{1}{2\sqrt{x}}$ ✓ C) $\frac{1}{3\sqrt{x}}$ D) $\frac{1}{\sqrt{x}}$

22. If f(x) = cotx, then the derivative of f(x) is

A) cosec $^2 x$ B) sec $^2 x$ C)- cosec $^2 x$ ✓ D) -sec $^2 x$

23. If f(x) = x e^x**, then the derivative of f(x) is**

A) x e^x + x ✓ B) e^x + x C) x- e^x + x D) x- x e^x

24. If f(x) = ($x^2+\frac{1}{x^2}$**)tanx , then the derivative of f(x) is**

A) ($x^2+\frac{1}{x^2}$)sec $^2 x$ B) ($x^2+\frac{1}{x^2}$)sec $^2 x$ +tanx(2x- $\frac{2}{x^2}$) ✓

C) ($x^2+\frac{1}{x^2}$)sec $^2 x$ +tanx(2x+ $\frac{2}{x^2}$) D) ($x^2-\frac{1}{x^2}$)sec $^2 x$ +tanx(2x- $\frac{2}{x^2}$)

25. If f(x) = $\frac{2x}{4+x^2}$ **, then the derivative of f(x) is**

A) $\frac{6-2x^2}{(4+x^2)}$ B) $\frac{8-2x^2}{(4+x^2)}$ C) $\frac{6-2x^2}{(4+x^2)^2}$ D) $\frac{8-2x^2}{(4+x^2)^2}$ ✓

26.If y = $(1-x^2)^{10}$**, then the derivative of y is**

A)-10x $(1-x^2)^9$ B) -30x $(1-x^2)^9$

C)-40x $(1-x^2)^9$ D) -20x $(1-x^2)^9$ ✓

27. If y = tan(sinx), then the derivative of y is

A)-cosx $\sec^2(\sin x)$ B) cosx $\sec^2(\sin x)$ ✓

C) sinx $\sec^2(\sin x)$ D) tanx $\sec^2(\sin x)$

28.The derivative of y = cosec $^{-1}$**x is**

A) $\frac{1}{x\sqrt{x^2-1)}}$ B) $-\frac{1}{x\sqrt{x^2-1)}}$ ✓ C) $\frac{1}{x\sqrt{x^2+1)}}$ D) $-\frac{1}{x\sqrt{x^2+1)}}$

29. If f(x) = $\frac{\sec x}{1+\tan x}$ **, then the derivative of f(x) is**

A) $\frac{\sec x(\tan x+1)}{(1+\tan x)^2}$ B) $\frac{\sec x(\tan x+1)}{(1-\tan x)^2}$

C) $\frac{\sec x(\tan x-1)}{(1+\tan x)^2}$ ✓ D) $\frac{\sec x(\tan x+1)}{(1-\tan x)^2}$

30. The derivative of y = x^x **is**

A) x^x (2+logx) B) x^x (1-logx)

C) x^x (1+logx) ✓ D) x^x (2-logx)

31.The derivative of y = sechx is

A) sechx tanhx B)cosechx tanhx

C) -sechx tanhx ✓ D)- cosechx tanhx

32.The equation of tangent at a given point (x_1,y_1) is

A) (x-x_1) = m (y-y_1) B) (y-y_1) = $\frac{-1}{m}(x-x_1)$

C) (y-y_1) $=\frac{1}{m}(x-x_1)$ D)(y-y_1) = m (x-x_1) ✓

33..The equation of normal at a given point (x_1,y_1) is

A) (x-x_1) = m (y-y_1) B) (y-y_1) $=\frac{-1}{m}(x-x_1)$ ✓

C) (y-y_1) $=\frac{1}{m}(x-x_1)$ D)(y-y_1) = m (x-x_1)

34. The equation of tangent line to the curve $y=9-x^2$ at the point (2,1) is

A) 4x-y = 9 B) 4x+y = 9 ✓ C) 4x+y = 6 D)3x+2y = 6

35.The equation of tangent line to the parabola curve $y=x^2$ at the point (1,1) is

A) 2x-y = 1 B) 2x+y = -1 C) 2x-y = -1 ✓ D)2x+2y = 1

36.The equation of tangent line to the curve $xy=c^2$ at the point (a,b) is

A) $\frac{x}{b}+\frac{y}{a}=1$ B) $\frac{x}{a}+\frac{y}{b}=1$ C) $\frac{x}{a}+\frac{y}{b}=2$ ✓ D) $\frac{x}{a}-\frac{y}{b}=2$

37.Which of the following statement is true?

Let c be a point in a domain D of a function f .Then $f^{'}(c)$ is the local maximum value if

A) $f(c)\le f(x)$ when x is near c B) $f(c)\le f(x)$ for all x in D

C) $f(c)\ge f(x)$ when x is near c ✓ D) $f(c)\ge f(x)$ for all x in D

38. Which of the following statement is true?

Let c be a point in a domain D of a function f .Then $f^{'}(c)$ is the Absolute minimum value if

A) $f(c)\le f(x)$ when x is near c B) $f(c)\le f(x)$ for all x in D ✓

C) $f(c)\ge f(x)$ when x is near c D) $f(c)\ge f(x)$ for all x in D

39.The critical numbers of the function f(x) = $5x^2+4x$ is

A) x = $\frac{4}{5}$ B) x = $\frac{2}{5}$ C) x = $-\frac{2}{5}$ ✓ D) x = $-\frac{4}{5}$

40. The critical point of the function f(x) = $2x^3 - 3x^2 - 36x$ **is**

A) x = 3, 2 B) x = -3,2 C) x = 3,-2 ✓ D) x = -3,-2

41.Suppose f^{11} **is continuous near c**

i)If $f^{'}(c)$ = 0 and f^{11} > 0, then f has a local maximum at c

ii) If $f^{'}(c)$ = 0 and f^{11} < 0, then f has a local minimum at c its According to

A) Rolle's theorem B) Concavity Test

C) First derivative Test D) Second derivative Test ✓

42.In which interval the function f(x) = $3x^4 - 4x^3 - 12x^2 + 5$ **is increasing**

A) x < -1 B) -1< x < 0 and x > 2 ✓

C) 0 < x < 2 D) $(-\infty, \infty)$

43. The critical point of the function f(x) = x+2sinx, $0 \le x \le 2\pi$

A) x = $\frac{5\pi}{3}, \frac{7\pi}{3}$ B) x = $\frac{2\pi}{3}, \frac{4\pi}{3}$ ✓

C) x = $\frac{\pi}{2}, \frac{3\pi}{2}$ D) x = $\frac{\pi}{3}, \frac{2\pi}{3}$

44.The Absolute minimum value of the function f(x) = $x^3 - 3x^2 + 1$ **is**

A)Minimum value f(4) = 17 B)Minimum value f(2) = 17

C)Minimum value f(2) = -3 ✓ D)Minimum value f(4) = -3

45.If f(2) =10.where $f^{'}(x) = x^2 f(x)$ **for all x, then the value of** $f^{11}(2)$ **is**

A) 100 B) 175 C) 165 D) 200 ✓

46.Afunction f is defined by f(x) = $\begin{cases} 1+x & if \ x < -1 \\ x^2 & if \ -1 \le x \le 1 \\ 2-x & if \ x \ge 1 \end{cases}$ **then the value of f(-2) is**

A) 0 B) 3 C) -1 ✓ D) 4

47. If a function *f* is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*) such that *f*(*a*) = *f*(*b*), then *f′*(*c*) = 0 for some *c* with $a \le c \le b$. it's according to

A)Langrange's mean value theorem B)sequeeze theorem

C)First derivative Test D)Rolle's theorem ✓

48.Find the constant 'c' if f(x) = $x^2 - 2x - 8$ **on [-1,3] By using rolle's theorem**

A) -8 B) 2 C) -1 D) 1 ✓

49.If $y = \sinh^{-1} x$ **then** $\frac{dy}{dx} =$

A) $\frac{1}{\sqrt{1+x^2}}$ ✓ B) $-\frac{1}{x\sqrt{x^2-1})}$ C) $-\frac{1}{\sqrt{x^2}+1}$ D) $-\frac{1}{x\sqrt{x^2}+1)}$

50.If $\lim_{x\to1}\frac{f(x)-8}{x-1} = 10$ then $\lim_{x\to1} f(x)$

A) 8 ✓ B) 11 C) 16 D) 10

51.For the function f(x) = sin(x)/x^2 How many points exist in the interval [0, 7π] Such that f'(c) = 0.

A) 8 B) 0 C) 7 D) 6 ✓

52.For second degree polynomial it is seen that the roots are equal. Then what is the relation between the Rolles point c and the root x?

A)c = x ✓ B) c = x^2 C) They are independent D) c = sin(x)

53.Rolle's Theorem tells about the

A) Existence of point c where derivative of a function becomes zero ✓
B) Existence of point c where derivative of a function is positive
C) Existence of point c where derivative of a function is negative
D) Existence of point c where derivative of a function is either positive or negative

54. Rolle's Theorem is a special case of

A) Lebniz Theorem B) Mean Value Theorem ✓
C) Taylor Series of a function D) Leibnit'x Theorem

55. Find the value of c(a point where slope of a tangent to curve is zero)

if f(x) = Sin(x) is continuous over interval [0,π] and differentiable over interval (0, π) and c ∈(0,π)

A) π B) π⁄2 ✓ C) π⁄6 D) π⁄4

56. f (x) = 3Sin(2x), is continuous over interval [0,π] and differentiable over interval (0,π) and c ∈(0,π)

A) π B) π⁄2 ✓ C) π⁄4 D) π⁄8

57. Evaluate $\lim_{x\to\infty}\frac{3x^5-4x^2+1}{-7x^5+x^2+2}$

A)-3/7 ✓ B)0 C)½ D)Undefined

58. Evaluate $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

A) e ✓ B)1 C)0 D)Undefined

59. A value of c for which the Mean value theorem holds for the function

f(x) = $\log_e x$ on the interval [1, 3] is

A) $2\log_3 e$ ✓ B) 12loge3 C) $\log_3 e$ D) $\log_e 3$

60. The value of c in Mean value theorem for f(x) = x(x – 2), x ∈ [1, 2] is

A) 3/2 ✓ B) 2/3 C) ½ (D) 5/2

61. The value of c in Rolle's theorem for the function, f(x) = sin 2x in [0, π/2] is

A) π/2 B) π/4 ✓ C) π/3 D) π/6

62. The value of c in Rolle's Theorem for the function f(x) = e^x sin x, x ∈ [0, π] is

A) π/6 B) π/4 C) π/2 D) 3π/4 ✓

63. The value of c in Mean value theorem for the function f(x) = x(x – 2),

x ∈ [1, 2] is

A) 32 ✓ B) 23 C) 12 D) 52

64. Find both the maximum and minimum values respectively of

$3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval [1, 4].

A) -63, 257 B) 257, -40 **C)** 257, -63 ✓ D) 63, -257

65.The equation of the normal to the curves y = sin x at (0, 0) is

A) x = 0 B) x + y = 0 ✓ C) y = 0 D) x – y = 0

66.Find all the points of local maxima and local minima of the function

$f(x) = (x - 1)^3 (x + 1)^2$

A) 1, -1, -1/5 ✓ B) 1, -1 C) 1, -1/5 D) -1, -1/5

67.If $y = x^3 + x^2 + x + 1$, then y

A) has a local minimum B) has a local maximum

C) neither has a local minimum nor local maximum ✓ D) None of these

69.Find the points of inflection of the function $f(x) = \sin 2x + x^2$ on the interval $0 \le x \le \pi /2$.

A)0, π /4 B)0, π /2 C) π /6 , 5π /6 D) π/ 12, 5π /12 ✓

70.Find the extreme values of the function & where they occur $f(x) = x^2 + 2x - 3$

A) Absolute minimum is -1 at x = 4. B) Absolute minimum is 1 at x = 4.

C) Absolute minimum is 1 at x = -4. D) Absolute minimum is -4 at x = -1. ✓

71.The maximum and minimum values of $f(x)=2x^3-24x+4$ is

A). max value= 17 , min value =-28 B) a. max value= 36 , min value =-28 ✓

C). max value= 36 , min value =-16 D). max value= 36 , min value =-15

72. Mean Value theorem is also known as

A) Rolle's Theorem B) Lagrange's Theorem ✓

C) Taylor Expansion D) Leibnitz's Theorem

73 .The point of inflection of $f(x)= x^3-6x^2+8x+5$ are

A)x=1 B) x= 2 ✓

C)x= -1 D) None of these

74.Value of $\lim_{x \to 0}(1+\sin(x))^{\mathrm{Cosec}(x)}$

A) e ✓ B) 0 C) 1 D) ∞

75. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at

A) (0, 1) B) (-1/2, 0) ✓ C) (2, 0) D) (0, 2)

76. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is

A) [-1, ∞] B) [-2, -1] ✓ C) [-∞, -2] D) [-1, 1]

77. Which of the following functions is decreasing on(0, π/2)?

A) sin 2x B) tan x C) cos x ✓ D) cos 3x

78. The function f (x) = $1 - x^3 - x^5$ is decreasing for

A) $1 < x < 5$ B) $x < 1$

C) $x > 1$ D) all values of x ✓

79. The function f(x) = x + cos x is

A) always increasing ✓ B) always decreasing

C) increasing for certain range of x D) None of these

80.The equation of the tangent to the curve $y = 4 + \sin^2 x$ at x = 0 is

A) y = 2 B)y = 3 C) y = 4 ✓ D) y = 6

81. Find the slope of $x^2y = 8$ at the point (2, 2)

A) 2 B)-1 C) -1/2 D) -2 ✓

82. Find the equation of the normal to $x^2 + y^2 = 5$ at the point (2, 1)

A) y = 2x B)x = 2y ✓ C) 2x + 3y = 3 D) x + y = 1

83. In the curve $2 + 12x - x^3$, find the critical points.

A) (2, 18) and (-2, -14) ✓ B) (2, 18) and (2, -14)

C)(-2, 18) and (2, -14) D)(-2, 18) and (-2, 14)

84. Locate the points of inflection of the curve $y = f(x) = x^2 e^{x.}$

A)$-2 \pm \sqrt{3}$ B) $2 \pm \sqrt{2}$ C) $-2 \pm \sqrt{2}$ ✓ D)$2 \pm \sqrt{3}$

85. What is the equation of the normal to the curve $x^2 + y^2 = 25$ at (4, 3)?

A) 5x + 3y = 0 B) 3x – 4y = 0 ✓ C) 3x + 4y = 0 D)5x – 3y = 0

86. The function $f(x) = x^3 - 6x^2 + 9x + 25$ has

A) a maxima at x =1 and a minimum at x=3 ✓

B) a maxima at x =3 and a minimum at x=1

C)No maxima butba minima at x= 1

D) a maxima at x =1 but no minima

87.The function f(x) = 3x(x-2) has a

A) minimum at x=1 ✓ B) maximum at x=1

C) minimum at x=2 D) minimum at x=2

88. If f(x) =$|x|$ in the interval [-1,1] then f(x)

A) Satisfied all the conditions of Rolle's theorem

B) Satisfied all the conditions of Mean value theorem

C) Does not Satisfied all the conditions of Mean value theorem ✓

D) None of these

89. $\lim_{x\to 0}\frac{\sin^2 x}{x}$ **is equal to**

A)0 ✓ B)∞ C)1 D)-1

90.The value of 'c' of the Mean value theorem for the function f(x) = x(x-2) when a=0, b=3/2 is

A)3/4 ✓ B)1/2 C)3/2 D)1/4

91. If f(x) = 3x4 -4x2 +5, then the interval for which f(x) satisfied all the condition of Rolle's Theorem is

A) [0, 2] B) [-1, 1] ✓ **C)** [-1,0] D) [1, 2]

92. The greatest and least value of f(x) = x^4- $8x^3$ + $22x^2$ - 24x +1 in [0, 2] are

A) 0, 8 B) 0, -8 C) 1,8 D) 1, -8 ✓

93. If a function is continuous at a point, then its first derivative

A) may or may not exist B) exists always

C) will not exist D) has a unique value ✓

94. The value of ξ in the mean value theorem of f(b) - f(a) = (b - a)f' (ξ) for f(x) = Ax^2 + bx + C in (a, b) is

A) b + a B) b – a C) $\frac{(b+a)}{2}$ ✓ D) $\frac{(b-a)}{2}$

95. **Which of the following is correct?**

A) f (a) is an extreme value of f(x) if f '(a)=0

B) If f (a) is an extreme value of f(x), then f '(a) = 0

C) If f '(a) = 0, then f (a) is an extreme value of f(x)

D) All of these ✓

96. If x + y = k, x > 0, y > 0, then xy is maximum when

A**)** x = ky B) kx = y C) **x = y** ✓ D**)** None of these

97. For second degree polynomial it is seen that the roots are equal. Then what is the relation between the Rolle's point c and the root x?

A) c = x ✓ B) $c = x^2$ c) They are independent d) c = sin(x)

98. Mean Value theorem is applicable to the

A) Functions differentiable in closed interval [a, b] and continuous in open interval (a, b)
B) Functions continuous in closed interval [a, b] only &having same value at point 'a' and 'b'
C) Functions continuous in closed interval [a, b] and differentiable in open interval (a, b) ✓
D) Functions differentiable in open interval (a, b) only &having same value at point 'a' &'b'

99. Mean Value Theorem tells about the

A) Existence of point c in a curve where slope of a tangent to curve is equal to the slope of line joining two points in which curve is continuous and differentiable ✓
B) Existence of point c in a curve where slope of a tangent to curve is equal to zero
C) Existence of point c in a curve where curve meets y axis
D) Existence of point c in a curve where curve meets x axis

100. Find the point c in the curve $f(x) = x^3 + x^2 + x + 1$ in the interval [0, 1] where slope of a tangent to a curve is equals to the slope of a line joining (0,1)

A) 0.64 B) 0.54 ✓ C) 0.44 D) 0.34

Maxima and Minima of functions of One Variable:

Definition:- Let x be a number in the domain D of a function f. Then $f(c)$ is the

↳ absolute maximum value of f on D if $\underline{f(c) \geq f(x)}$ for all x in D.

↳ absolute minimum value of f on D if $\underline{f(c) \leq f(x)}$ for all x in D.

Definition:- The number $f(c)$ is a

↳ local maximum value of f if $\underline{f(c) \geq f(x)}$ when x is near c.

↳ local Minimum value of f if $\underline{f(c) \leq f(x)}$ when x is near c.

Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Fermat's theorem:- If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$

Definition:-

A Cricital number of a function f is a number c in the domain of f such that either $f'(c)=0$ (or) $f'(c)$ does not exist.

Definition:-

If f has a local maximum or minimum at c, then c is a Cricital number of f.

---

1. Find the Critical Value of the function $f(x) = 5x^2+4x$.

Soln: Given: $f(x) = 5x^2+4x$

Critical Values of f occur at $f'(x)=0$

$$f'(x) = 5(2x)+4(1)$$
$$= 10x+4$$
$$\therefore f'(x) = 0$$
$$\Rightarrow 10x+4=0$$
$$\Rightarrow 10x = -4$$
$$\Rightarrow x = \frac{-4}{10} = -2/5$$

$\therefore$ The Critical Values are $x = -2/5$

---

2. Find the Critical Values of $g(x) = 2x^3 - 3x^2 - 36x$.

Soln. Given: $g(x) = 2x^3 - 3x^2 - 36x$

$$g'(x) = 2(3x^2) - 3(2x) - 36(1)$$
$$= 6x^2 - 6x - 36$$

$\therefore g'(x) = 0$

$\Rightarrow 6x^2 - 6x - 36 = 0$

$\Rightarrow 6(x^2 - x - 6) = 0$

$\therefore x = -2, 3$

-6
-3 2
-1

$\therefore$ The Critical Values are $x = -2, 3$

o

3. Find the Critical Values of $f(x) = x^2 - 32\sqrt{x}$.

Soln. Given: $f(x) = x^2 - 32\sqrt{x}$

$$f'(x) = 2x - \cancel{32}^{16}\left(\frac{1}{\cancel{2}\sqrt{x}}\right)$$
$$= 2x - 16\left(1/\sqrt{x}\right)$$
$$= 2\left(x - 8/\sqrt{x}\right)$$

$\therefore f'(x) = 0$

$\Rightarrow 2\left(x - 8/\sqrt{x}\right) = 0$

$\Rightarrow x - 8/\sqrt{x} = 0$

$\Rightarrow x = 8/\sqrt{x} \Rightarrow x^2 = 8^2/x$

$\Rightarrow x^3 = 64 \Rightarrow x = 4, -2 \pm i(3.464)$

The Critical Value are $x = 4$ (only real numbers)

o

4. Find the Critical Values of $f(x) = 2\cos\theta + \sin^2\theta$.

Soln. Given: $f(x) = 2\cos\theta + \sin^2\theta$.

$f'(x) = -2\sin\theta + 2\sin\theta\cos\theta$ $\quad$ ($\sin^2\theta = 2\sin\theta\cos\theta$

$\therefore f'(x) = 0$

$\Rightarrow -2\sin\theta + 2\sin\theta\cos\theta = 0$

$-2\sin\theta[1 - \cos\theta] = 0$

$\Rightarrow \theta = n\pi$, n is an integer.

The Critical Value $= n\pi$, n is an integer.

---

5. Find the absolute Maximum and minimum Value of $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x \le 4$ (or) $[-\frac{1}{2}, 4]$.

Soln. Given: $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x \le 4$.

$f'(x) = 3x^2 - 3(2x)$

$= 3x^2 - 6x$

$\therefore f'(x) = 0$

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$x = 0$, $x - 2 = 0$

$x = 2$

$\therefore$ The Critical Values are $x = 0, 2$

x:	$-\frac{1}{2}$	0	1	2	3	4
$f(x)$:	$\frac{1}{8}$	1	-1	-3	1	17

$f(4) = 17$ is the absolute maximum Value of f.

$f(2) = -3$ is the absolute minimum Value of f.

---

6. Find the absolute max. and Mini. Value of

$f(x) = \log(x^2+x+1)$, $[-1, 1]$.

Soln. Given: $f(x) = \log(x^2+x+1)$, $[-1, 1]$

$$f'(x) = \frac{1}{x^2+x+1}(2x+1)$$

$$\therefore f'(x) = 0$$

$$\Rightarrow \frac{2x+1}{x^2+x+1} = 0$$

$$\Rightarrow 2x+1 = 0$$

$$\Rightarrow 2x = -1 \Rightarrow x = -\tfrac{1}{2}.$$

The critical value $x = -\tfrac{1}{2}$

x:	-1	$-\tfrac{1}{2}$	0	1
$f(x)$:	0	0.75	0	0.477

$f(-\tfrac{1}{2}) = 0.75$ is the absolute maximum Value of f

$f(-1) = 0$ is the absolute minimum Value of f.

O

7. Find the absolute Max. and Min. Value of

$f(x) = 2\cos x + \sin 2x$, $[0, \pi/2]$.

Soln. Given: $f(x) = 2\cos x + \sin 2x$

$$f'(x) = -2\sin x + 2\cos 2x$$

$$\therefore f'(x) = 0$$

$\Rightarrow -2\sin x + 2\cos 2x = 0$

$\Rightarrow 2\cos 2x = 2\sin x$

$\Rightarrow \cos 2x = \sin x$

$\Rightarrow x = \pi/6$

$\therefore$ The Critical Value $= \pi/6$.

$f(\pi/6) = 3/2\sqrt{3}$ is the absolute Max. Value of f.

$f(\pi/2) = 0$ is the absolute mini. Value of f.

---

8. For the function $f(x) = 2x^3 + 3x^2 - 36x$.
(i) Find the Critical Points
(ii) Find the intervals on which it is increasing or decreasing.
(iii) Find the local maximum and Minimum Values of f.
(iv) Find the intervals of Concavity and the inflection Points.

Soln. (i) Given $f(x) = 2x^3 + 3x^2 - 36x$.

$f'(x) = 2(3x^2) + 3(2x) - 36(1)$

$= 6x^2 + 6x - 36$

$\therefore f'(x) = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

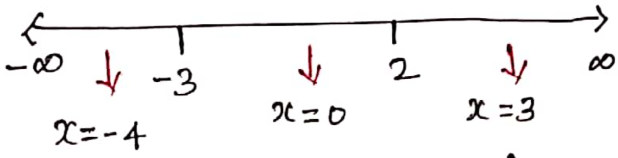
$\Rightarrow 6(x^2 + x - 6) = 0$

$\Rightarrow x^2 + x - 6 = 0$

-6
3 -2
1

$\therefore x = -3, 2$

$\therefore$ The critical points are $x = -3, 2$



Take.

$f'(x) = 6(-4)^2 + 6(-4) - 36$
$= 36$

$f'(0) = 6(0)^2 + 6(0) - 36$
$= -36$

$f'(3) = 6(3)^2 + 6(3) - 36$
$= 36$

(ii)

Interval	Sign of $f'(x)$	Behavior of $f(x)$
$(-\infty, -3)$	+	increasing
$(-3, 2)$	−	decreasing
$(2, \infty)$	+	increasing

local max. at $x = -3$

local mini. at $x = 2$.

(iii) Here $f(-3) = 81$ is a local Maximum Value at $x = -3$.

$f(2) = -44$ is a local minimum value at $x = 2$.

(iv) $f''(x) = 6(2x) + 6(1)$

$= 12x + 6$

$\therefore f''(x) = 0$

$\Rightarrow 12x + 6 = 0$

$\Rightarrow 12x = -6$

$\Rightarrow x = -6/12 = -1/2$

$\therefore x = -1/2$

$-\infty \quad \downarrow \quad -\frac{1}{2} \quad \downarrow \quad \infty$

$x = -1 \qquad x = 0$

$f''(x) = 12x + 6$

$f(-1) = -6$, $f''(0) = 6$

Interval	Sign of $f''(x)$	Behavior $f(x)$
$(-\infty, -\frac{1}{2})$	$-$	Concave downward
$(-\frac{1}{2}, \infty)$	$+$	Concave upward.

Here inflection point at $\left(-\frac{1}{2}, f(-\frac{1}{2})\right) = \left(-\frac{1}{2}, \frac{37}{2}\right)$.

$\therefore f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + 3(-\frac{1}{2})^2 - 36(-\frac{1}{2})$

$= 2(-\frac{1}{8}) + 3(\frac{1}{4}) + \frac{36}{2}$

$= -\frac{1}{4} + \frac{3}{4} + 18$

$= \frac{-1+3}{4} + 18$

$= \frac{2}{4} + 18$

$= \frac{1}{2} + 18$

$\therefore f(-\frac{1}{2}) = \frac{1+36}{2} = \frac{37}{2}$.

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OMR, Chennai - 119.

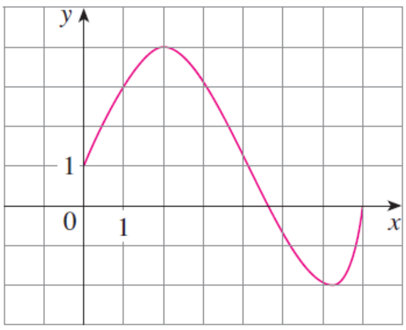
MULTIPLE CHOICE QUESTIONS

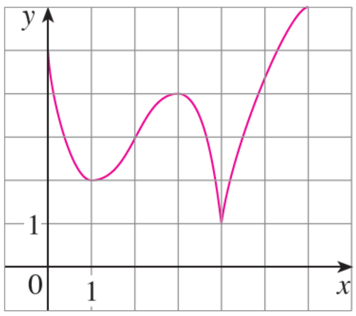
Subject Name: Engineering Mathematics-I **Subject Code: MA8151**

Year: I Year (All Branches) **Semester: I**

Assignment – II

UNIT I – DIFFERENTIAL CALCULUS

Differential Calculus-Functions	
1	The domain of the function $f(x) = 2x - 1$ is a) $[0,\infty)$ b) $(-\infty,\infty)$ c) $(-\infty,0]$ d) None of these
2	Which one of the following is the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$. a) $[-2,3]$ b) $[2,3]$ c) $(-\infty,3]$ d) $(-\infty,\infty)$
3	The domain and range of the function $f(x) = \sqrt{4-x^2}$ are a) Domain: $-2 \le x \le 2$ & Range: $0 \le y \le 2$ b) Domain: $0 \le x \le 2$ & Range: $0 \le y \le 2$ c) Domain: $-2 \le x \le 2$ & Range: $-2 \le y \le 2$ d) Domain: $0 \le x \le \infty$ & Range: $0 \le y \le \infty$
4	From the graphical representation of a function $y = f(x)$, the domain and range are  a) Domain: $[0,\infty)$ & Range: $[0,\infty)$ b) Domain: $[0,7]$ & Range: $[-2,4]$ c) Domain: $(-\infty,\infty)$ & Range: $(-\infty,\infty)$ d) None of these
5	Which one of the following function has the domain $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$?

	a) $f(x)=\dfrac{2x^3-5}{x^2+x-6}$ b) $f(x)=\sqrt{3-x}-\sqrt{1+x}$ c) $f(x)=\dfrac{x+4}{x^2-9}$ d) $f(x)=\dfrac{2x^3}{x^2+9}$
6	Which of the following function is not an even function? a) x^2+1 b) $\cos x$ c) x^4+x^2 d) x^3+x
7	Which of the following statements are true: (i) The function $x+x^2$ is an even function (ii) The function e^{x^2} is odd function a) (i) is true and (ii) is false b) (i) is false and (ii) is true c) Both are true d) Both are false
8	Which one of the following is an increasing function over the real line R? a) $f(x)=x+1$ b) $f(x)=x^2$ c) $f(x)=-x$ d) None of the these
9	Which one of the following is decreasing function over $(-\infty,\infty)$? a) $y=\|x\|$ b) $y=\cos x$ c) $y=\sin x$ d) $y=-x$
10	From the graphical representation of a function $y=f(x)$, The open intervals on which $f(x)$ is increasing are  a) (1,3), (4,6) b) (0,1), (3,4) c) (1,4), (5,6) d) None of these
	Limit of a Function
11	The $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^2+4}$ is a) 1 b) 0 c) -1 d) 2
12	Which of the following is the value of $\lim\limits_{x\to 0}\dfrac{x}{x}$? a) 1 b) 0 c) -1 d) ∞
13	The $\lim\limits_{x\to\infty}\dfrac{4-x^2}{x^2-1}$ is a) 1 b) 0 c) -1 d) ∞
14	The value of $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$ a) 0 b) 1 c) ∞ d) does not exist

15	The $\lim_{x\to 0}\frac{1-\cos x}{x}$ is equal to a) 1 b) 0 c) -1 d) 2
16	The value of $\lim_{x\to\infty} x^2 \sin\frac{1}{x}$ a) 0 b) 1 c) ∞ d) does not exist
17	If $f(x)\le g(x)\le h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)=L.$ a) Sandwich theorem b) Fermat's theorem c) Extreme value theorem d) Mean value theorem
18	Which of the following is the value of $\lim_{x\to 0}\frac{\sin x}{x^2+3x}$? a) 1 b) 0 c) ∞ d) $\frac{1}{3}$
19	Which of the following is the limit of a function yields limit value 5? a) $\lim_{x\to\infty}\frac{e^x-1}{x^5}$ b) $\lim_{x\to 0}\frac{e^{5x}-1}{x^5}$ c) $\lim_{x\to 0}\frac{e^{5x}-1}{x}$ d) $\lim_{x\to 0}\frac{5e^{5x}-1}{x}$
20	Which of the following is the value of $\lim_{x\to 0}\frac{\|x\|}{x}$? a) 0 b) 1 c) ∞ d) does not exist
	Continuity of a Function
21	A function $f(x)$ is said to be continuous at x_0 for $x\in R$, if (i) $f(x_0)$ is defined (ii) $\lim_{x\to x_0} f(x)$ exists $(i.e)\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)$ (iii) $\lim_{x\to x_0} f(x)= f(x_0)$ a) (i) and (ii) holds b) only (ii) holds c) only (iii) holds d) (i), (ii) and (iii) holds
22	A function $f(x)$ is said to be not continuous at x_0 for $x\in R$, if (i) $f(x)$ is not defined at x_0 (ii) $\lim_{x\to x_0} f(x)$ does not exists (iii) $\lim_{x\to x_0} f(x)\ne f(x_0)$ a) all the Above holds b) only (ii) holds c) only (iii) holds d) only (i) and (ii) holds
23	A function $f(x)$ is said to be continuous for $x\in R$, if a) it is continuous at $x=0$ b) differentiable at $x=0$ c) continuous at two points d) differentiable for $x\in R$

24	The function $f(x)=\dfrac{1}{x-5}$ for $x\in R$ a) continuous at $x=5$ b) not continuous at $x=5$ c) continuous everywhere d) nowhere continuous
25	The function $f(x)=\dfrac{x^2-16}{x-4},\ x\neq 4,$ for $x\in R$ is a) continuous everywhere b) not continuous at $x=4$ c) continuous at $x=4$ d) nowhere continuous
26	The function $f(x)=2x^2+1$ for $x\in R$ is a) continuous only at x=0 b) not continuous at $x=0$ c) continuous on $(-\infty,\infty)$ d) nowhere continuous
27	Which of the following function has discontinuity at $x=0$ a) e^x b) $e^{\frac{1}{x}}$ c) $\sin x$ d) $x+5$
28	Which of the following function has discontinuity at $x=3$ a) $x-3$ b) cos x c) $\dfrac{1}{x-3}$ d) x^3+3
29	A function $f(x)=\begin{cases} x^2\ , & x<0 \\ kx, & x>0 \end{cases}$ is a) continuous everywhere b) nowhere continuous c) continuous at $x=0$ d) not continuous at $x=0$
30	Given functions $f(x)=\dfrac{x^2-4}{x-2}$ and $g(x)=x+2,\ x\in R$. Then which of the following is correct? a) f is continuous at $x=2$, g is continuous at $x=2$ b) f is continuous at $x=2, g$ is not continuous at $x=2$ c) f is not continuous at $x=2, g$ is continuous everywhere. d) f is not continuous at $x=2, g$ is not continuous at $x=2$
31	A function $f(x)=\begin{cases} \dfrac{sin 5x}{3x}+cos\,x & if\ x\neq 0 \\ k & if\ x=0 \end{cases}$ is continuous at $x=0$ for a) $k=1$ b) $k=\dfrac{8}{3}$ c) $k=-1$ d) $k=\dfrac{4}{3}$
32	A function $f(x)=\begin{cases} \dfrac{sin\,x}{x}+cos\,x & if\ x\neq 0 \\ 2k & if\ x=0 \end{cases}$ is continuous at $x=0$ for a) $k=1$ b) $k=2$ c) $k=\dfrac{1}{2}$ d) $k=\dfrac{3}{2}$
33	Find the value of k, so that the function $f(x)=\begin{cases} kx^2 & \text{if } x<1 \\ 4 & \text{if } x\geq 1 \end{cases}$ is continuous

	a) $k=2$ b) $k=3$ c) $k=4$ c) $k=-2$
34	For what value of the constant k, is the function $f(x)$ continuous at x=0 $f(x)=\begin{cases} \frac{kx}{\lvert x\rvert}, & x<0 \\ 3, & x\geq 0 \end{cases}$ a) $k=3$ b) $k=-3$ c) $k=0$ c) $k=1$
35	Which of the following is true for the continuity of the function $f(x)$ is defined by $f(x)=\begin{cases} 1+x, & if\ x\leq -2 \\ 2-x, & if\ -2<x\leq 2. \\ 2x-4, & if\ 2<x \end{cases}$ a) continuous at $x=-2,2$ b) continuous $x=-2$ *and discontinuous at* $x=2$ c) continuous at $x=2$ *and discontinuous at* $x=-2$ d) discontinuous at x = −2, 2
36	The functions $\lvert x\rvert$ is a) continuous at $x=0$ b) discontinuous at $x=0$ c) continuous everywhere d) nowhere continuous
37	The function $f(x)=\lvert x-a\rvert$ is a) continuous everywhere b) nowhere continuous c) discontinuous at $x=a$ d) continuous at $x=a$
38	If $f(x)=\begin{cases} xe^{-\left(\frac{1}{\lvert x\rvert}+\frac{1}{x}\right)} & x\neq 0 \\ 0 & if\ x=0 \end{cases}$ is then $f(x)$ is a) continuous everywhere b) nowhere continuous c) continuous at $x=0$ d) discontinuous at $x=0$
Derivative	
39	The functions $\lvert x\rvert$ is a) differentiable at $x=0$ b) not differentiable at $x=0$ c) Everywhere continuous d) discontinuous at $x=0$
40	The function $f(x)=\lvert x-a\rvert$ is a) differentiable at $x=a$ b) not differentiable at $x=a$ c) Everywhere continuous d) discontinuous at $x=a$
41	If $f(x)=\begin{cases} xe^{-\left(\frac{1}{\lvert x\rvert}+\frac{1}{x}\right)} & x\neq 0 \\ 0 & if\ x=0 \end{cases}$ is then $f(x)$ is a) differentiable at $x=0$ b) not differentiable at $x=0$ c) Everywhere continuous d) discontinuous at $x=0$
42	If $y=\cos\left(x^2\right)$ then its derivative is

	a) $y'=-\sin\left(x^2\right)$ b) $y'=2x\sin\left(x^2\right)$ c) $y'=-2x\sin\left(x^2\right)$ d) $y'=-2\sin\left(x^2\right)$
43	If $f(x)=\frac{1}{\sqrt{x}}$, then derivative of $f(x)$ is a) $f'(x)=\frac{1}{2x\sqrt{x}}$ b) $f'(x)=\frac{1}{2\sqrt{x}}$ c) $f'(x)=\frac{2}{\sqrt{x}}$ d) $f'(x)=-\frac{1}{2x\sqrt{x}}$
44	The derivative of a function *f* at a point *x* is a) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ b) $f'(x)=\lim_{h\to 0}\frac{f(x-h)+f(x)}{h}$ c) $f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$ d) $f'(x)=\lim_{h\to \infty}\frac{f(x+h)-f(x)}{h}$
Tangent and Normal	
45	The tangent line to the hyperbola $y=\frac{1}{x}$ at the point (1,1) is a) $x+y+2=0.$ b) $x+y-2=0.$ c) $x+y-4=0.$ d) $x+y-1=0.$
46	Which of the following is the equation of the tangent line to the curve $y=f(x)$ at (x_1,y_1)? a) $y+y_1=m(x-x_1)$ b) $y+y_1=m(x+x_1)$ c) $y-y_1=m(x-x_1)$ d) $y-y_1=\frac{1}{m}(x-x_1)$
47	What are the points on the curve $y=x^4-6x^2+4$ where the tangent line is horizontal? a) $(0,4),\left(\sqrt{3},-5\right),\left(-\sqrt{3},-5\right).$ b) $(4,0),\left(\sqrt{3},-5\right),\left(-\sqrt{3},-5\right).$ c) $(0,4),\left(\sqrt{3},5\right),\left(-\sqrt{3},-5\right).$ d) $(0,4),\left(5,\sqrt{3}\right),\left(-\sqrt{3},-5\right).$
48	Which of the following is the equation of the normal line to the curve $y=f(x)$ at (x_1,y_1)? a) $y-y_1=\frac{1}{m}(x-x_1)$ b) $y+y_1=m(x+x_1)$ c) $y-y_1=m(x-x_1)$ d) $y-y_1=-\frac{1}{m}(x-x_1)$
49	The equation of the normal line to the curve $y=x^4+2e^x$ at the point (0,2) is a) $x+y+2=0.$ b) $x+2y-4=0.$ c) $x+2y=-4$ d) $x-2y=-4$
Rules of Derivatives	
50	Which one of the following is the first derivative of $y=e^x\sin x$? a) $e^x\sin x$ b) $e^x(\cos x+\sin x)$ c) $e^x\sin x-\cos x$ d) None of these
51	The first derivative of $y=e^{e^x}$ is a) $e^{e^x}e^x$ b) e^{e^x} c) $e^{e^x}\log x$ d) None of these
52	The derivative of the function $f(x)=x^2(x+1)$ a) $y'=2x$ b) $y'=3x^2$ c) $y'=2x+1$ d) $y'=3x^2+2x$
53	Which one of the following is the derivative of the function $\left(\frac{2+z}{z^3}\right)$

	a) $\frac{6+2z}{z^4}$ b) $\frac{1}{3z^2}$ c) $\frac{-2z-6}{z^4}$ d) $\frac{3z^2-(z+2)}{z^6}$、
54	The derivative of $f(x)=\frac{x}{\sin x}$ is a) $\frac{\sin x-\cos x}{\sin^2 x}$ b) $\frac{\sin x-x\cos x}{\sin^2 x}$ c) $\frac{\sin x+x\cos x}{\sin^2 x}$ d) $\frac{\sin x+x\cos x}{\sin x}$
55	The derivative of the given function $y=\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right)$ a) $\frac{1}{2}$ b) $\frac{3}{2}$ c) 1 d) 3
56	The derivative of the given function $xy=2x+y$ is a) $\frac{2+y}{x+1}$ b) $\frac{2-y}{x+1}$ c) $\frac{2-y}{x-1}$ d) None of these
57	The $\frac{dy}{dx}$ of the function $y=\log x^3+5\log x^2$ a) $\frac{13}{x}$ b) 3log x – 10log x c) $\frac{-13}{x}$ d) $\log\left(\frac{3x^2}{10x}\right)$
58	The $\frac{dy}{dx}$ of the function $y=(\sin x)^x$ a) $(\sin x)^x[x\cot x-\log\sin x]$ b) $x\cot x+\log\sin x$ c) $(\sin x)^x[x\cot x+\log\sin x]$ $d)$ $\sin x[x\cot x+\log\sin x]$
59	What is $f'(6)$ if $f(x)=x\sqrt{2x-3}$ a) 6 b) 5 c) 7 d) 8
60	Consider the velocity of a car $V=2t^3+3t^2-2t$ what is the acceleration for $t=2$? a) 28 b) 30 c) 16 d) 34
61	The derivative of the function $y=\sqrt{x+\sqrt{x+\sqrt{x}+.....\infty}}$ a) $\frac{1}{2y+1}$ b) $\frac{1}{2y-1}$ c) $\frac{-1}{2y-1}$ d) $\frac{-1}{2y+1}$
62	What is the second derivative of the function $y=x^4+2x^3-10x^2+12$ $a)$ $4x^3+6x^2-20x$ $b)$ $12x^2+6x-20$ $c)$ $12x^2+12x-20$ $d)$ $12x^2+12x$
63	Which one of the following is the first derivative of the function $x=a\cos^3 t$ and $y=a\sin^3 t$ a) $\tan t$ b) $\sin t$ c) $-\sin t$ d) $-\tan t$
64	If $y=\ln\left(e^x\ln x\right)$ then what is y' a) 1 b) $1+\frac{1}{x}$ c) $\frac{1}{x}$ d) $\ln e^x$

65	What is $\frac{d^2y}{dx^2}$ for $x = ct$ and $y = \frac{c}{t}$ a) $\frac{1}{ct^3}$ b) $\frac{2}{ct^3}$ c) ct^3 d) $-\frac{2}{ct^3}$
66	What is the differentiation of $e^{\sin^{-1}x}$ with respect to $\sin^{-1}x$ a) $e^{\sin^{-1}x}$ b) $e^{\sin x}$ c) $\sin x$ d) $\sin^{-1}x$
Derivatives of Trigonometric Functions	
67	The derivative of $f(x) = -\cos ec\left(\frac{x}{2}\right)$ is a) $sec^2\left(\frac{x}{2}\right)$ b) $2\cos ec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)$ c) $\frac{1}{2}\cos ec\left(\frac{x}{2}\right)\cot\left(\frac{x}{2}\right)$ d) $-\cos ec^2\left(\frac{x}{2}\right)$
68	What is the derivative of $f(x) = -\sin x$? a) $-\sin x$ b) $-\cos x$ c) $\cos x$ d) $\sin x$
69	If the rate of change of derivative of $y = f(x)$ is $-\frac{1}{1+x^2}$, then y is a) $\cot x$ b) $-\cos ec^2 x$ c) $\tan x$ d) $\sec^2 x$
70	If $y(x) = \sec 4x$ then $y'(x)$ is a) $\cos ec4x$ b) $4\sec 4x \tan 4x$ c) $-4\cos ec4x \cot 4x$ d) $4\tan 4x$
71	What is $f'(x)$ given $f(x) = 4\tan\left(\frac{1}{2} - x\right)$? a) $4\sec^2\left(\frac{1}{2} - x\right)$ b) $-4\cos ec^2\left(\frac{1}{2} - x\right)$ c) $-4\sec^2\left(\frac{1}{2} - x\right)$ d) $4\cos ec^2\left(\frac{1}{2} - x\right)$
72	What is the derivative of $y = \cos(\sin x)$? a) $-\sin(\sin x)\cos x$ b) $\sin(\sin x)\cos x$ c) $\cos(\sin x)\sin x$ d) $\sin(\sin x)\sin x$
73	If $f(x) = \sin^3(2x)$ then $f'(x)$ is a) $3\cos 2x$ b) $3\sin 2x - 2$ c) $3\sin^2(2x)\cos 2x$ d) $6\sin^2(2x)\cos 2x$
74	What is the derivative of $\arctan\sqrt{e^x}$ at $x = 0$? a) $-\frac{1}{4}$ b) $\frac{1}{4}$ c) 0 d) 1
75	The derivative of $\sin\sqrt[3]{3x}$ is a) $(3x)^{\frac{2}{3}}\cos(3x)^{\frac{1}{3}}$ b) $3\cos\sqrt[3]{3x}$ c) $\frac{\cos(3x)^{\frac{1}{3}}}{(3x)^{\frac{2}{3}}}$ d) $\sin(3x)^{-\frac{2}{3}}\cos 3x$
76	What is the derivative of $y = \sec(x^2 + 2)$? a) $2x\cos(x^2+2)$ b) $-\cos(x^2+2)\cot(x^2+2)$ c) $2x\sec(x^2+2)\tan(x^2+2)$ d) $\cos(x^2+2)$

77	What is y' if $y = \arcsin(\cos x)$? a) -1 b) -2 c) 1 d) 2
78	The derivative of $f(x) = \sinh x - \cosh x$ is a) $\sinh x + \cosh x$ b) $\cosh 2x$ c) $\sinh 2x$ d) $\cosh x - \sinh x$
79	What is the derivative of $y = \dfrac{\tan x}{2x-3}$? a) $\dfrac{dy}{dx} = \dfrac{(2x-3)\sec x \tan x - 2\tan x}{(2x-3)^2}$ b) $\dfrac{dy}{dx} = \dfrac{(2x-3)\sec^2 x - 2\tan x}{(2x-3)^2}$ c) $\dfrac{dy}{dx} = \dfrac{\sec^2 x - 2\tan x}{(2x-3)^2}$ d) $\dfrac{dy}{dx} = \dfrac{(2x-3)\operatorname{cosec}^2 x - 2\tan x}{(2x-3)^2}$
80	If $f(x) = \cosh\left(\dfrac{x}{3}\right)$ then $f'(x)$ is a) $\sinh\left(\dfrac{x}{3}\right)$ b) $\dfrac{1}{3}\sinh\left(\dfrac{x}{3}\right)$ c) $-\sinh\left(\dfrac{x}{3}\right)$ d) $\cosh\left(\dfrac{x}{3}\right)$
81	The derivative of $f(x) = \sinh 3x - \cosh 5x$ is a) $\sinh 3x - \cosh 5x$ b) $3\cosh 3x + 5\sinh 5x$ c) $3\cosh 3x - 5\sinh 5x$ d) $\sinh 3x + \cosh 5x$
Absolute Maximum and Minimum & Local Maximum and Minimum	
82	A critical point of a function $f(x)$ is a point c in the domain of $f(x)$ such that either a) $f(c) = 0$ or $f(c)$ does not exist b) $f'(c) = 0$ or $f'(c)$ does not exist c) $f(c) = 0$ or $f'(c)$ does not exist d) None of these
83	It is given that at $x = 1$, the function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0,2]$. Then the value of a is a) 100 b) 120 c) 200 d) None of these
84	The critical number of the function $f(x) = 6x^2 + 3x$ is a) $\dfrac{2}{3}$ b) $\dfrac{1}{4}$ c) $-\dfrac{1}{4}$ d) None of these
85	If $\dfrac{2}{5}$ is the critical number of a function $f(x)$, then $f(x)$ is a) $5x^2 - 4x$ b) $5x^2 + 4x$ c) $10x + 4$ d) None of these
86	The critical number of the function $f(x) = x^3 - x^2 - x$ is a) $1, 3$ b) $-1, \dfrac{1}{3}$ c) $1, -\dfrac{1}{3}$ d) $-1, -3$
87	The absolute maximum value of $f(x) = (x^2 - 1)^3$ on $[-1,1]$ is a) 1 b) 0 c) 2 d) None of these
88	The absolute minimum value of $y = x^3 - 3x^2 + 1$ in $0 \le x \le 4$ is

	a) 1 b) 17 c) 3 d) -3
89	The absolute maximum value of $y = x^3 - 3x + 2$ in $0 \le x \le 2$ is a) 4 b) 6 c) 2 d) 0
90	If $f(x) = -(x-1)^2 + 10$, then the maximum value is a) 12 b) 10 c) 11 d) None of these
91	The extreme values if any of the function given by $f(x) = \sin 2x + 5$ is a) Maximum value 4 and minimum value 6 b) Maximum value 6 and minimum value -6 c) Maximum value 6 and minimum value 4 d) Maximum value 4 and minimum value 4
92	The function $f(x) = x^3 + x^2 + x + 1$ as a) Maximum value b) Minimum value c) Extreme values d) No Extreme values
93	The stationary point of the function $f(x) = x^x, x > 0$ is a) $\frac{1}{e}$ b) e c) $-\frac{1}{e}$ d) $-e$
94	The function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing in a) $x < -1$ b) $-1 < x < 0$ c) $0 < x < 2$ d) None of these
95	The function $f(x) = x + 2\sin x$ is decreasing in a) $0 < x < \frac{2\pi}{3}$ b) $\frac{2\pi}{3} < x < \frac{4\pi}{3}$ c) $\frac{4\pi}{3} < x < 2\pi$ d) None of these
96	If $f(x) = \sin x - \cos x$, then interval in which function is decreasing in $0 \le x \le 2\pi$, is a) $\left[\frac{5\pi}{6}, \frac{3\pi}{4}\right]$ b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ c) $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ d) None of these
97	Which of the following statement is not correct? a) The function $f(x) = 4x + 3, x \in R$ is an increasing function b) The function $f(x) = \log(\cos x)$ is increasing function for $\left[0, \frac{\pi}{2}\right]$ c) The function $y = 4x - 9$ is increasing for all $x \in R$ d) The function $f(x) = \sin x$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$
98	The local minimum value of the function $f(x) = \sin^4 x + \cos^4 x$, $0 < x < \frac{\pi}{2}$ is a) $-\frac{1}{2}$ b) 2 c) $\frac{1}{2}$ d) -2
99	Let $f(x) = 2x^3 + 3x^2 - 36x$. Then the interval of concave downward is

	a) $x<-\frac{1}{2}$ b) $x>-\frac{1}{2}$ c) both $x<-\frac{1}{2}$ and $x>-\frac{1}{2}$ d) None of these
100	Let $f(x)=2+2x^2-x^4$. Then the interval of concave upward is a) $x<-\frac{1}{\sqrt{3}}$ b) $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$ c) $x>\frac{1}{\sqrt{3}}$ d) None of these
101	Let $f(x)=x^4-4x^3$. Then the point of inflection is a) (0, 0) and (2, −16) b) (0, 1) and (2, 16) c) (1, 1) and (1, −16) d) (1, 1) and (2, 16)
102	Let $f(x)=x^2(6-x)$. Then the point of inflection is a) (16, 2) b) (1, 16) c) (−2, 16) d) (2, 16)
103	The points of inflection of the function $f(x)=x^4-12x^3+6x-9$ on the interval $-2\le x\le 10$ is a) $x=0,6$ b) $x=0,-6$ c) $x=\pm 6$ d) $x=\pm 12$
104	The points of inflection of the function $f(x)=\sin 2x+x^2$ on the interval $0\le x\le \frac{\pi}{2}$ is a) $0,\frac{\pi}{4}$ b) $0,\frac{\pi}{2}$ c) $\frac{\pi}{6},\frac{5\pi}{6}$ d) $\frac{\pi}{12},\frac{5\pi}{12}$

UNIT – I

Differential Calculus

Representation of function: Function:-

* A function f from a set D to a set E is a rule that assigns a unique element $f(x) \in E$ to each element $x \in D$.

* The set D of all possible input values is called the domain of the function.

* The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Real-Valued functions:-

A function whose domain and co-domain are subsets of the set of all real numbers is known as real-valued function.

Explicit functions:-

If x and y be so related that y can be expressed explicitly in terms x, then y is called explicit function of x.

Example:- $y = x^2 - 4x + 2$.

Implicit functions:

If x and y be so related that y cannot be expressed explicity in terms of x, then y is called implicit function of x.

Example:- $x^3+y^3-3xy=0$.

Domain, Co-domain, range and image:

Let $f: A \to B$ then set A is called the domain of the function set B is called co-domain.

The set of all the images of all the elements of A under the function f is called the range of f and is denoted by $f(A)$.

The range of f is $f(A) = \{f(x) : x \in A\}$.

clearly $f(A) \subseteq B$.

If $x \in A$, $y \in B$ and $y = f(x)$, then y is called the image of x under f.

Graph of functions:

If f is a function with domain D, then its graph is the set of ordered pairs $\{(x, f(x)) / x \in D\}$.

Piecewise-defined functions:-

The functions are described by using different formula's on different parts of its domain, such function are called piecewise defined functions.

The Vertical line test for a function:

A Curve in the xy-plane is the graph of a function of x if and only if no Vertical line intersects the Curve more than once.

Even function:-

If a function $y = f(x)$ is an even function of x if $\boxed{f(-x) = f(x)}$

Odd function:-

If a function $y = f(x)$ is an odd function of x if $\boxed{f(-x) = -f(x)}$.

1. Find the domain and range and sketch the graph of the function $f(x) = x^2$.

Soln: Given: $f(x) = x^2$

$\Rightarrow y = x^2$

The given equation is a parabola

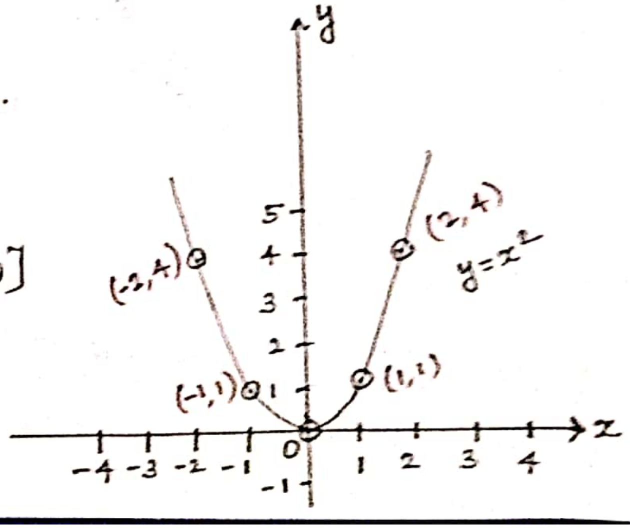
Domain (x):	$-\infty$	...	-2	-1	0	1	2	...	∞
Range (y)	∞	...	4	1	0	1	4	...	∞

Here $x, y \in R$

The Domain is $(-\infty, \infty)$

The Range is $[0, \infty)$

$[\because x^2 \geq 0]$



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2. Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0. \end{cases}$$

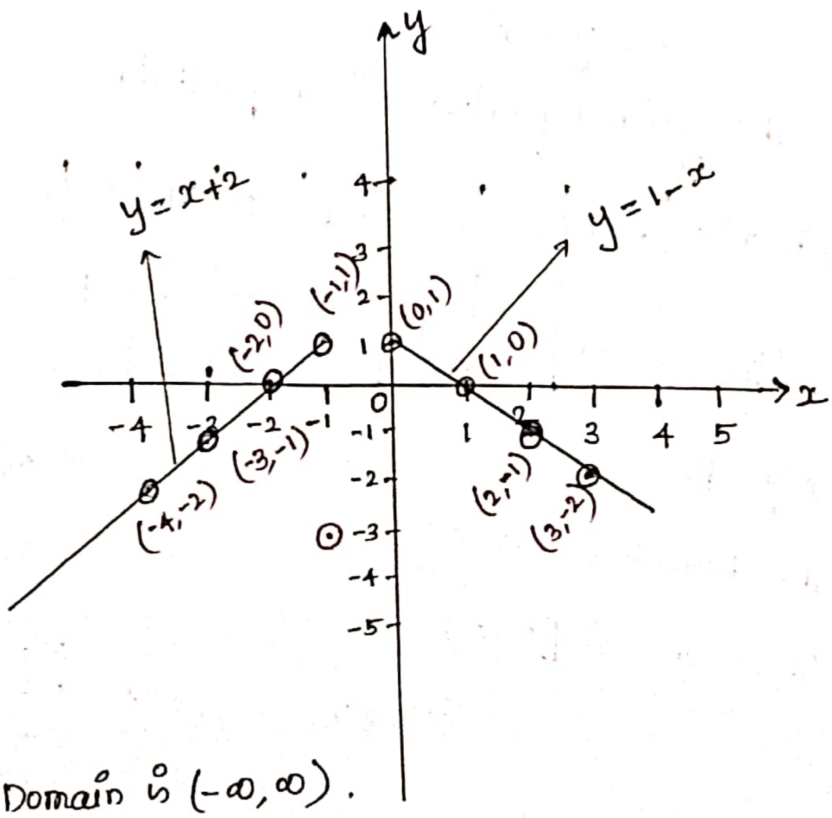
Soln: Given: $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0. \end{cases}$

(ie) $y = x+2, \ x < 0$

$y = 1-x, \ x \geq 0.$

$x < 0$	-1	-2	-3	-4	...
$y = x+2$	1	0	-1	-2	...

$x \geq 0$	0	1	2	3	...
$y = 1-x$	1	0	-1	-2	...



The Domain is $(-\infty, \infty)$.

3. Find the domain and range the function $f(x) = \frac{4}{3-x}$.

Soln. Given: $y = f(x) = \frac{4}{3-x}$

(ie) $y = \frac{4}{3-x}$ [$\because$ Divison by zero is not allowed]

Domain (x)	$-\infty$	...	-2	-1	0	1	2	...	3	...	4	...	∞
Range (y)	0	...	$4/5$	1	$4/3$	2	4	... ∞	ND	$-\infty$	-4	...	0

ND $\rightarrow$ Not defined

Here $x, y \in R$,

So, the domain is $(-\infty, 3) \cup (3, \infty)$

Range is $(0, \infty) \cup (-\infty, 0)$

o

4. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$.

Soln. Given: $f(x) = \frac{x+4}{x^2-9}$

$\Rightarrow y = \frac{x+4}{x^2-9}$

$\Rightarrow x^2 - 9 = 0$

$\Rightarrow x^2 = 9$

$\therefore x = \pm 3$

So the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

o

5. Find the domain of the function $f(x) = \sqrt{x+2}$

Soln: Given: $f(x) = \sqrt{x+2}$

$\Rightarrow y = \sqrt{x+2}$

$x+2 \geq 0$ [$\because$ Square root of a negative number is not defined].

$x \geq -2$

So, the domain is $[-2, \infty)$.

Even function: $\boxed{f(-x) = f(x)}$

Odd function: $\boxed{f(-x) = -f(x)}$.

6. Determine whether each of the following function is even, odd or neither even nor odd.

(i) $f(x) = x^2+1$

Soln: Given: $f(x) = x^2+1$

$f(-x) = (-x)^2+1$

$= x^2+1$

$\boxed{f(-x) = f(x)}$

$\therefore f(x)$ is an even function

(ii) $f(x) = x\cos x$.

Soln: **Given:** $f(x) = x\cos x$

$[\because \cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

$f(-x) = (-x)\cos(-x)$

$= -x\cos x$

$\boxed{f(-x) = -f(x)}$

$\therefore f(x)$ is an odd function.

o

(iii) $f(x) = x+1$

Soln: **Given:** $f(x) = x+1$

$f(-x) = -x+1$

$\boxed{f(-x) \neq f(x)}$

$\therefore f(x)$ is neither even nor odd function.

o

7. Evaluate $f(x) = 4+3x-x^2$, $\frac{f(3+h)-f(3)}{h}$.

Soln: Let $f(x) = 4+3x-x^2$

$f(3+h) = 4+3(3+h)-(3+h)^2$

$= 4+9+3h-(9+h^2+6h)$

$= 4+9+3h-9-h^2-6h$

$= 4-3h-h^2$

$$f(3) = 4 + 3(3) - (3)^2$$

$$= 4 + \cancel{9} - \cancel{9}$$

$$f(3) = 4$$

$$\therefore \frac{f(3+h) - f(3)}{h} = \frac{1}{h}\left[\cancel{4} - 3h - h^2 - \cancel{4}\right]$$

$$= \frac{1}{h}\left[-3h - h^2\right]$$

$$= -\frac{h}{h}\left[3 + h\right]$$

$$= -(3+h)$$

H.W

1. Find the domain of the function $f(x) = \frac{1}{x^2 - x}$.

2. Evaluate $2x^2 - 5x + 1$, $\frac{f(a+h) - f(a)}{h}$, $h \neq 0$.

3. If $f(x) = x^3 + x$ is odd or even function.

MA 8151 - ENGINEERING MATHEMATICS-I

UNIT-I

Differential Calculus.

1.1 Representation of functions:

Function:

A function f from a set D to a set E is a rule that assigns a unique element $f(x) \in E$ to each element $x \in D$.

The set D of all possible input values is called the domain of the function.

The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Real-valued functions:

A function whose domain and co-domain are subsets of the set of all real numbers is known as real-valued function.

Explicit functions:

If x and y be so related that y can be expressed explicity in terms x, then y is called explicit function of x.

Ex: $y = x^2 - 4x + 2$.

Implicit functions:

If x and y be so related that y cannot be expressed explicitly in terms of x, then y is called implicit function of x.

Eg: $x^3+y^3-3xy=0$.

Domain, co-domain, range and image:

Let $f: A \to B$ then

set A is called the domain of the function

set B is called co-domain

The set of all the images of all the elements of A under the function f is called the range of f and is denoted by $f(A)$.

The range of f is $f(A) = \{f(x) : x \in A\}$,

Clearly $f(A) \subseteq B$.

If $x \in A$, $y \in B$ and $y = f(x)$, then y is called the image of x under f.

Graph of functions:

If f is a function with domain D, then its graph is the set of ordered pairs $\{(x, f(x)) / x \in D\}$.

Piecewise - defined functions:

The functions are described by using different formula's on different parts of its domain. Such functions are called piecewise-d.f.

The vertical line test for a function:

A curve in the xy-plane is the graph of a function of x iff no vertical line intersects the curve more than once.

Even function and odd function:

If a function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ and odd function of x if $f(-x) = -f(x)$ for every number x in its domain.

Problem based on the domain and range and sketch the graph of the function:

1. Find the domain and range and sketch the graph of the function $f(x) = x^2$

Solution: Given: $f(x) = x^2$

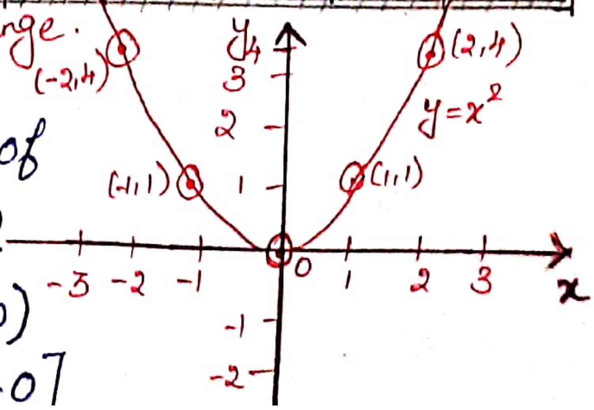
$\Rightarrow y = x^2$.

$\Rightarrow$ Given equation is a parabola. Domain

Domain(x)	$-\infty$	...	-2	-1	0	1	2	....	∞
Range (y)	∞	...	4	1	0	1	4	....	∞

Range.

Here $x, y \in \mathbb{R}$.



So the domain of f is the set of all real numbers $\mathbb{R}$. ie, $(-\infty, \infty)$.

The graph of the range is $[0, \infty)$

$[\because x^2 \geq 0]$

2. Find the domain and sketch the graph of the function $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0. \end{cases}$

Solution:

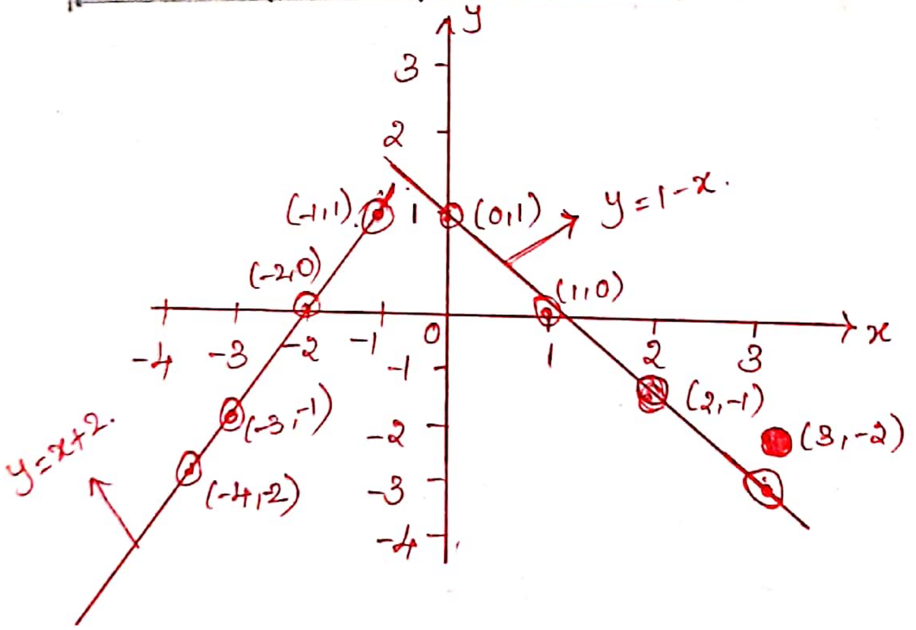
Given: $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0. \end{cases}$

ie, $y = x+2$, $x<0$

$y = 1-x$, $x \geq 0$.

$x<0$	-1	-2	-3	-4	....
$y=x+2$	1	0	-1	-2	....

$x \geq 0$	0	1	2	3	...
$y=1-x$	1	0	-1	-2	...



$\therefore$ The domain is $(-\infty, \infty)$.

3. Find the domain and the range of each function $f(x) = \frac{4}{3-x}$.

Solution: Given : $f(x) = \frac{4}{3-x}$

$\Rightarrow y = \frac{4}{3-x}$ (divison by zero is not allowed)

Domain (x)	$-\infty$	...	-2	-1	0	1	2	...	3	...	4	...	∞
Range (y)	0	...	4/5	1	4/3	2	4	...∞	N.D	$-\infty$...	-4	...	0

Here $x, y \in R$, N.D $\rightarrow$ Not defined.

So, the domain is $(-\infty, 3) \cup (3, \infty)$

and the range is $(-\infty, 0) \cup (0, \infty)$.

4. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$.

Solution: Given: $f(x) = \frac{x+4}{x^2-9}$

$\Rightarrow y = \frac{x+4}{x^2-9}$

$\Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3.$

So the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

5. Find the domain of the function $f(x) = \sqrt{x+2}$.

Solution: Given : $f(x) = \sqrt{x+2}$

$\Rightarrow y = \sqrt{x+2}$.

$x + 2 \geq 0$ [$\because$ square root of a negative number is not defined]

$x \geq -2$.

So the domain is $[-2, \infty)$.

Even function : $f(-x) = f(x)$

Odd function : $f(-x) = -f(x)$.

* **Determine whether each of the function is even, odd or neither even nor odd.**

1. $f(x) = x^2+1$

Given : $f(x) = x^2+1$

$$f(-x) = (-x)^2+1$$

$$= x^2+1$$

$$f(-x) = f(x)$$

$\therefore$ $f(x)$ is an even function.

2. $f(x) = x \cos x$.

$\left[\because \cos(-\theta) = \cos\theta, \ \sin(-\theta) = -\sin\theta\right]$

Given: $f(x) = x \cos x$

$$f(-x) = (-x)\cos(-x)$$

$$= -x \cos x$$

$$f(-x) = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

3. $f(x) = x+1$

Given : $f(x) = x+1$

$$f(-x) = -x+1$$

$$f(-x) \neq f(x)$$

$$f(-x) = -(x-1)$$

$$f(-x) \neq -f(x)$$

$\therefore$ $f(x)$ is neither even nor odd function.

Evaluate the difference quotient for the given function

1. $f(x) = 4+3x-x^2$, $\frac{f(3+h)-f(3)}{h}$

Solution: Given: $f(x) = 4+3x-x^2$

$f(3+h) = 4+3(3+h)-(3+h)^2 = 4+9+3h-(9+h^2+6h)$

$= -h^2-3h+4$

$f(3) = 4+3(3)-(3)^2 = 4+9-9 = 4$

$\frac{f(3+h)-f(3)}{h} = \frac{-h^2-3h+4-4}{h} = \frac{-h^2-3h}{h} = -3-h.$

H.W

1. $f(x) = x^3$, $\frac{f(a+h)-f(a)}{h}$ Ans: $h^2+3ah+3a^2$

2. $f(x) = 2x^2-5x+1$, $\frac{f(a+h)-f(a)}{h}$, $h \neq 0$.

3. $f(x) = 3x^2-x+2$, find $f(a)$, $f(-a)$, $f(a+1)$, $f(a^2)$ and $f(a+h)$ Ans: i) $3a^2-a+2$ ii) $3a^2+a+2$ iii) $3a^2+5a+4$ iv) $3a^4-a^2+2$, v) $3a^2+6ah+3h^2-a-h+2$.

4. $f(x) = 1-x^4$, $f(x) = x^5+x$, Check it even or odd.

5. Sketch the graph and find the domain and range of the function $f(x) = 2x-1$.

6. Find the domain of the function $f(x) = \frac{1}{x^2-x}$.

$x^2-x = 0.$

$x(x-1) = 0$

$x = 0, x = 1$

$(-\infty, 0)$ $(0, 1)$ $(1, \infty)$.

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Definition:

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^-} f(x) = L \ \& \ \lim_{x \to a^+} f(x) = L.$$

Infinite Limits:

Let f be a function defined on both sides of a, except possibly at a itself.
Then $\lim_{x \to a} f(x) = \infty$ mean that $f(x)$ can be arbitrarily large by taking x sufficiently close to a, but not equal to a.

Definition:

The line $x = a$ is called a vertical asymptote of the curve $f(x)$ if atleast one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \quad \lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \quad \lim_{x \to a^+} f(x) = -\infty \quad \lim_{x \to a^-} f(x) = -\infty.$$

Determine the infinite Limit:

1. $\lim_{x \to 1} \dfrac{2-x}{(x-1)^2}$

Solution: Given: $\lim_{x \to 1} \dfrac{2-x}{(x-1)^2}$

$$= \frac{2-1}{(1-1)^2} = \frac{1}{0}$$

$$\lim_{x \to 1} \frac{2-x}{(x-1)^2} = \infty.$$

2. $\lim\limits_{x \to -3^+} \left(\frac{x+2}{x+3} \right)$

NOTE:
1) +ve, +ve = ∞
2) -ve -ve = ∞
3) +ve, -ve = $-\infty$
4) -ve, +ve = $-\infty$
5) +ve = ∞
6) -ve = $-\infty$.

Solution: Given: $\lim\limits_{x \to -3^+} \left(\frac{x+2}{x+3} \right)$

$x \to -3^+ \Rightarrow x$ is close to -3 but larger than -3.

Nr = $x+2$ becomes negative

Dr = $x+3$ becomes positive

Let $x = -2.9$

Nr = $(-2.9+2 = -0.9 = -ve)$

Dr = $(-2.9+3 = 0.1 = +ve)$

$\therefore \lim\limits_{x \to -3^+} \frac{x+2}{x+3} = -\infty.$

3. $\lim\limits_{x \to (2\pi)^-} x \operatorname{cosec} x.$

Solution: Given: $\lim\limits_{x \to (2\pi)^-} x \operatorname{cosec} x = \lim\limits_{x \to (2\pi)^-} \frac{x}{\sin x}$

$x \to (2\pi)^- \Rightarrow x$ is close to 2π but smaller than 2π

Nr = x becomes positive

Dr = $\sin x$ becomes negative $[\because \sin 290^\circ = -ve]$

$\therefore \lim\limits_{x \to (2\pi)^-} x \operatorname{cosec} x = -\infty.$

4. $\lim\limits_{x \to 3^+} \log(x^2 - 9)$

Solution:

Given: $\lim\limits_{x \to 3^+} \log(x^2 - 9)$

$x \to 3^+ \Rightarrow x$ is close to 3 but larger than 3

$\log(x^2 - 9)$ becomes negative $[\because \log[(3.1)^2 - 9] = -ve]$

$\therefore \lim\limits_{x \to 3^+} \log(x^2 - 9) = -\infty.$

5. $\lim\limits_{x \to 0^+} \left(\frac{1}{x} - \log x \right) = \infty$ (+ve)

6. $\lim\limits_{x \to -3^-} \left(\frac{x+2}{x+3} \right) = \infty$ (-ve, -ve)

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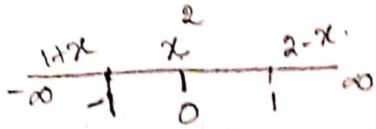
7. Sketch the graph of the function

$$f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1. \end{cases}$$

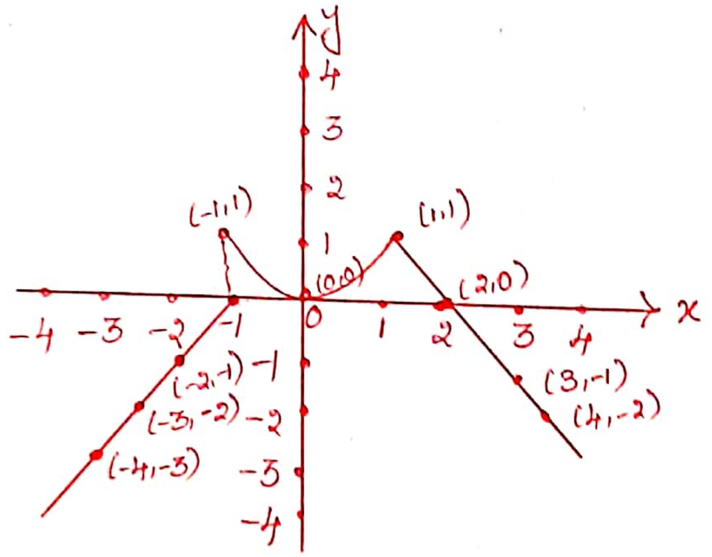
[Jan : 2018, 2017]

and use it to determine the values of 'a' for which $\lim_{x \to a} f(x)$ exists.



Solution:

	$1+x ; x < -1$			$x^2 ; -1 \le x \le 1$			$2 - x^2, x \ge 1$			
x	-2	-3	-4	-1	0	1	1	2	3	4
$f(x)$	-1	-2	-3	1	0	1	1	0	-1	-2



At $x = -1$,

$$f(-1^-) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (1+x) = 1-1 = 0$$

$$f(-1) = \lim_{x \to -1} f(x) = \lim_{x \to -1} x^2 = (-1)^2 = 1$$

$$f(-1^+) = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 = (-1)^2 = 1$$

Here, $f(-1^-) \ne f(-1) = f(-1^+)$

$\therefore$ f is discontinuous at $x = -1$.

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At $x=1$

$$f(1^-) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = (1)^2 = 1$$

$$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = (1)^2 = 1.$$

$$f(1^+) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1.$$

Here, $f(1^-) = f(1) = f(1^+)$

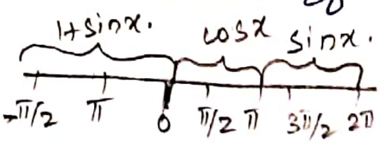
$\therefore$ f is continuous at $x=1$.

Hence, $\lim_{x\to a} f(x)$ exists for all 'a' except at $a=-1$.

	0	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

H.W.

8. Sketch the graph of the function $f(x) = \begin{cases} 1+\sin x, & x<0 \\ \cos x, & 0\le x\le \pi \\ \sin x, & x>\pi \end{cases}$ and use it to determine the value of 'a' for which $\lim_{x\to a} f(x)$ exists.



$\cos\pi = -1$

$\sin\pi = 0$

$\sin 0 = 0$

$\cos 0 = 1$

Eliminating Zero Denominators Algebraically

If the denominator is zero, cancelling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c.

1. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Solution: Given: $\lim_{x\to 1} \frac{x^2-1}{x-1}$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1) = 1+1 = 2.$$

$$\therefore \lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

2. $\lim_{x \to 1} \frac{x^4-1}{x^3-1}$

$\left[\because (a^2-b^2) = (a+b)(a-b) \quad (a^3-b^3) = (a-b)(a^2+ab+b^2)\right]$

Solution: Given: $\lim_{x \to 1} \frac{x^4-1}{x^3-1}$

$$= \lim_{x \to 1} \frac{(x^2+1)(x^2-1)}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{(x^2+1)(x+1)}{(x^2+x+1)} = \frac{(2)(2)}{3} = \frac{4}{3}$$

$$\therefore \lim_{x \to 1} \frac{x^4-1}{x^3-1} = \frac{4}{3}.$$

H.W

3. $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$: Ans : 3.

$\frac{x^2+x-2}{x^2-x} = \frac{x^2+x-2+x-x}{x(x-1)} = \frac{x^2+2x-2-x}{x(x-1)} = \frac{x(x+2)-1(x+2)}{x(x-1)} = 3/1 = \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} = 3.$

4. If $\lim_{x \to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x \to 1} f(x)$

$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

Solution: Given: $\lim_{x \to 1} \frac{f(x)-8}{x-1} = 10$

$$\Rightarrow \lim_{x \to 1} (f(x)-8)\left(\frac{1}{x-1}\right) = 10$$

$$\Rightarrow \lim_{x \to 1} (f(x)-8) \cdot \lim_{x \to 1} \left(\frac{1}{x-1}\right) = 10.$$

$$\Rightarrow \lim_{x\to 1} f(x) - 8 = 10 \cdot \lim_{x\to 1} (x-1) = 10(1-1) = 0$$

$$\therefore \lim_{x\to 1} f(x) = 8.$$

5. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, Find the following limits.

a) $\lim_{x\to 0} f(x) = 0$. b) $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$ when x is near 'a'

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

1. Use the squeeze theorem, find the value of $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution: Given: $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

W.K.T $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$$\Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} (-x^2) = \lim_{x\to 0} (x^2) = 0$$

$$\Rightarrow \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \quad [\because \text{By S.T}].$$

2. $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$

Solution: Given: $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$.

W.K.T, $-1 \le \cos\left(\frac{2}{x}\right) \le 1.$

$$\Rightarrow -x^4 \le x^4 \cos(2/x) \le x^4$$

$$\lim_{x\to 0} (-x^4) = \lim_{x\to 0} x^4 = 0$$

$$\Rightarrow \lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0 \quad [\text{By S.T}].$$

H.W.

3. $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

4. $\lim_{x\to 0} \sqrt{x^3+x^2}\, \sin\left(\frac{\pi}{x}\right).$

Special Limits:

1. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$ for all rational values of n.

2. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ θ is measured in radians.

3. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ for all rational values of n.

1. **Evaluate:** $\lim_{x\to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2}$ [U.Q Jan: 2016]

Solution: Given: $\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\pi/2} \frac{2\cos^2 x}{(\pi-2x)^2}$

$$= \lim_{x\to\pi/2} \frac{2\sin^2(\pi/2 - x)}{2^2(\pi/2 - x)^2} = \lim_{x\to\pi/2} \frac{1}{2}\left[\frac{\sin(\pi/2 - x)}{\pi/2 - x}\right]^2$$

$$= \lim_{(x-\pi/2)\to 0} \frac{1}{2}\left[\frac{\sin(x-\pi/2)}{(x-\pi/2)}\right]^2 \quad \text{Let } \theta = x - \pi/2,$$

$$= \lim_{\theta\to 0} \frac{1}{2}\left(\frac{\sin\theta}{\theta}\right)^2 = \frac{1}{2}\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right]^2 = \frac{1}{2}(1)^2$$

$$\left[\because \frac{\cos 0}{1} = \frac{1}{1} = 1\right]$$

$$= 1/2.$$

2. H.W. Find $\lim_{x\to 0} (1+x)^{1/x}$. $[\because \text{put } x = 1/n]$.

1.3 Continuity:

Continuous:

A function 'f' is continuous at a number 'a' if $\lim_{x \to a} f(x) = f(a)$

Discontinuous:

A function 'f' is discontinuous at a number 'a' if $\lim_{x \to a} f(x) \neq f(a)$.

Right Limit:

A function 'f' is continuous from the right at 'a' if $\lim_{x \to a^+} f(x) = f(a)$.

Left Limit: A function 'f' is continuous from the left at 'a' if $\lim_{x \to a^-} f(x) = f(a)$.

a) Locate the discontinuity of the function:

1. Explain the function $f(x) = \begin{cases} \cos x & , \text{ if } x < 0 \\ 0 & , \text{ if } x = 0. \\ 1 - x^2 & , \text{ if } x > 0 \end{cases}$, $a = 0$ is discontinuous at 'a'.

Solution: Given: $f(x) = \begin{cases} \cos x & , x < 0 \\ 0 & , x = 0 \\ 1 - x^2 & , x > 0. \end{cases}$

W.K.T $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \cos x = \cos 0 = 1$$

$$f(0) = 0$$

$$\therefore \lim_{x \to 0} f(x) \neq f(0).$$

$\therefore f(x)$ is discontinuous at 'a'.

2. Locate the discontinuous of the function $y = \frac{1}{1+e^{1/x}}$.

Solution: **Given:** $f(x) = \frac{1}{1+e^{1/x}}$.

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{1}{1+e^{-1/h}} = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{1}{1+e^{1/h}} = 0$$

$$f(0) = \frac{1}{1-e^{1/0}} = 0$$

$e^{1/0}$, e^{∞}

$\frac{1}{1+\infty} = \frac{1}{\infty} = 0$, $e^{0} = 1$, $e^{-\infty} = 0$, $e^{\infty} = \infty$, $\frac{1}{0} = \infty$.

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So $f(x)$ is discontinuous at $x = 0$.

3. Explain the function is continuous at 2.

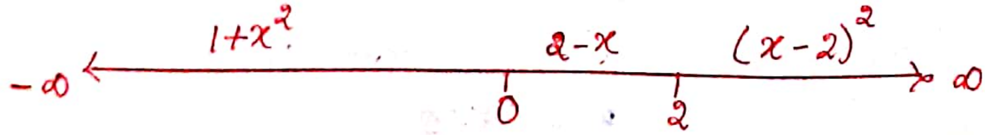
a) $f(x) = \frac{x^3-8}{x^2-4}$ [3] b) $f(x) = \frac{x^2-7x+10}{x-2}$. [-3]

4. Find the domain where the function f is continuous. Also find the numbers at which the function f is discontinuous, where

[U.Q - 2016, 2015]

$$f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2. \end{cases}$$

Solution:



At $x = 0$, $(2-x) = 2$

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x^2) = 1 \rightarrow ①$$

$$f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 + x^2) = 1 \rightarrow ②$$

$$f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x) = 2 \rightarrow ③$$

From ①, ②, ③ we get,

$$f(0^-) = f(0) \neq f(0^+).$$

So f is continuous on the left at $x = 0$

f is discontinuous on the right at $x = 0$.

Hence, f is discontinuous at $x = 0$.

At $x = 2$,

$$f(2) = \lim_{x \to 2} f(x) = \lim_{x \to 2} (2 - x) = 0 \rightarrow ①$$

$$f(2^-) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0 \rightarrow ②$$

$$f(2^+) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2)^2 = 0 \rightarrow ③$$

From ①, ②, ③ we get,

$$f(2^-) = f(2) = f(2^+).$$

Hence f is continuous at $x = 2$.

The domain of f is $(-\infty, 0) \cup (0, \infty)$.

5. Find the numbers that at which f is discontinuous, At which of numbers is f Continuous from the right from the left or neither? when

$$f(x) = \begin{cases} x + 2, & x < 0 \\ e^x, & 0 \leq x \leq 1 \\ 2 - x, & x > 1. \end{cases}$$

$\left.\begin{array}{l} f(0)^- = 2 \\ f(0) = 1 \\ f(0)^+ = 1 \end{array}\right\}$ dis at $x = 0$, $\left.\begin{array}{l} f(1)^- = e^1 = e \\ f(1) = e^1 = e \\ f(1)^+ = 2 - 1 = 1 \end{array}\right\}$ dis at $x = 1$.

$\therefore f$ is dis- at $x = 0$, $x = 1$.

$\therefore f$ is cont- at $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

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b) function is continuous in a given interval: 10

1. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Solution:

Given: $f(x) = 1 - \sqrt{1-x^2}$

$$\lim_{x \to -1^+} f(x) = 1 = f(-1); \quad \lim_{x \to 1^-} f(x) = 1 = f(1).$$

So, f is continuous from the right at -1

f is continuous from the left at 1

$\therefore$ f is continuous on $[-1, 1]$.

2. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3 f(x) + f(x) g(x)] = 36$. Find $f(2)$.

Solution:

Given: $\lim_{x \to 2} [3 f(x) + f(x) g(x)] = 36, \quad g(2) = 6.$

$$\Rightarrow 3 \lim_{x \to 2} f(x) + \lim_{x \to 2} f(x) g(x) = 36.$$

$$3 f(2) + f(2) \cdot g(2) = 36$$

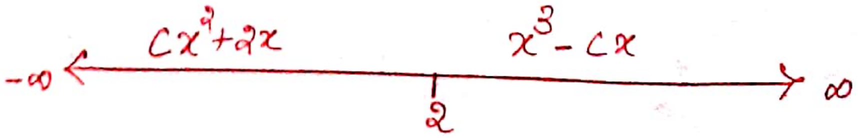
$$\Rightarrow f(2) [3+6] = 36 \quad \Rightarrow f(2)(9) = 36$$

$$f(2) = 4.$$

3. For what value of the constant 'c' is the function 'f' continuous at $(-\infty, \infty)$. [Jan : 2018]

$$f(x) = \begin{cases} cx^2 + 2x, & x < 2 \\ x^3 - cx, & x \geq 2. \end{cases}$$

Solution:



At $x=2$, Given: f is continuous.

$$\Rightarrow f(2^-) = f(2) = f(2^+) \rightarrow (1)$$

$$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left[x^3 - cx\right] = 8-2c. \rightarrow (a)$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left[cx^2+2x\right] = 4c+4 \rightarrow (b)$$

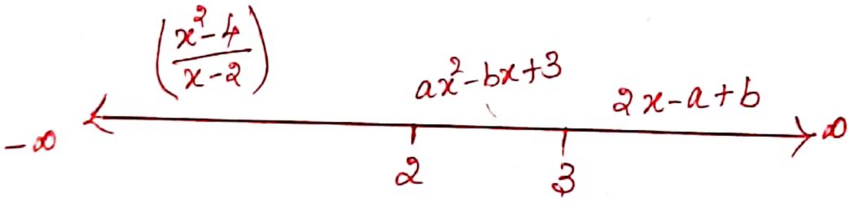
$$(1) \Rightarrow 8-2c = 4c+4 \quad [\text{by } (a) \;\&\; (b)]$$

$$6c = 4$$

$$c = 2/3.$$

4. If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x<2 \\ ax^2-bx+3, & 2\le x<3 \\ 2x-a+b, & x\geq 3 \end{cases}$ is continuous for all real x, find the values of a and b.

Solution:



At $x=2$, Given: $f(2^-) = f(2) = f(2^+) \rightarrow (1)$

$$f(2)^+ = \lim_{x\to 2} f(x) = \lim_{x\to 2}(ax^2-bx+3) = 4a-2b+3 \rightarrow (a)$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^-}(x+2) = 4 \rightarrow (b)$$

$$(1) \Rightarrow 4a-2b+3 = 4$$

$$\Rightarrow 4a-2b = 1 \rightarrow (c)$$

At $x=3$, Given: $f(3^-) = f(3) = f(3^+) \rightarrow (2)$

$$f(3) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(2x-a+b) = 6-a+b \rightarrow (d)$$

$$f(3^-) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(ax^2-bx+3) = 9a-3b+3 \rightarrow (e)$$

② $\Rightarrow$ $6-a+b = 9a-3b+3$

$\Rightarrow 10a-4b=3 \rightarrow$ ⓕ

Solving ⓔ & ⓕ weget,

$$4a-2b=1$$
$$10a-4b=3$$
$$a=\frac{1}{2},\ b=\frac{1}{2}.$$

5. Show that f is continuous on $(-\infty,\infty)$.

$$f(x)=\begin{cases}\sin x, & x<\pi/4\\ \cos x, & x\geq \pi/4\end{cases}$$

Solution:

At $x=\pi/4$ To prove: $f\left(\frac{\pi}{4}^-\right)=f\left(\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}^+\right) \rightarrow$ Ⓐ

$$f(\pi/4)=\lim_{x\to\pi/4} f(x)=\lim_{x\to\pi/4}\cos x=\cos 45^\circ=\frac{1}{\sqrt{2}} \rightarrow ①$$

$$f(\pi/4^-)=\lim_{x\to\pi/4^-} f(x)=\lim_{x\to\pi/4^-}\sin x=\sin 45^\circ=\frac{1}{\sqrt{2}} \rightarrow ②$$

$$f\left(\frac{\pi}{4}^+\right)=\lim_{x\to\frac{\pi}{4}^+} f(x)=\lim_{x\to\pi/4^+}\cos x=\cos 45^\circ=\frac{1}{\sqrt{2}} \rightarrow ③$$

From ① & ② & ③ weget,

$$f(\pi/4^-)=f(\pi/4)=f(\pi/4^+)$$

$\therefore f$ is continuous on $(-\infty,\infty)$.

6. Show that f is continuous on $(-\infty,\infty)$, $f(x)=\begin{cases}x^2, & x<1\\ \sqrt{x}, & x\geq 1.\end{cases}$

Ans:

To prove: $f(1^-)=f(1)=f(1^+)$

Ans: 1.

$\therefore f$ is continuous on $(-\infty,\infty)$.

1.4 Derivatives:

Tangent Line:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists.

Derivative:

The derivative of a function f at a number 'a', denoted by $f'(a)$ is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

NOTE:

The equation of the tangent line is,

$$(y - y_1) = m(x - x_1).$$

(or) $y - f(a) = f'(a)(x - a)$.

$f(x) = y = g(x)$
$f'(x) = y' = g'(x)$;
$m = f'(x) = y'$

1. Find an equation of the tangent line to the curve at the given point $y = \sqrt{x}$, $(1,1)$.

(1) **Solution:** **Given:** $y = f(x) = \sqrt{x}$, $(1,1)$

$$y' = f'(x) = \frac{1}{2\sqrt{x}}$$

Slope $m = f'(x) = (y')_{(1,1)} = \frac{1}{2}$.

The equation of the tangent line at $(1,1)$ is,

$$y - y_1 = m(x - x_1).$$

$$y - 1 = \frac{1}{2}(x - 1). \Rightarrow 2y - 2 = x - 1$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(13)

2. Find the slope of the tangent line to the parabola
(2) $y = 4x - x^2$ at $(1,3)$.

Solution: Given: $y = 4x - x^2$

$$f'(x) = 4 - 2x$$
$$m = f'(1) = 4 - 2$$
$$m = 2.$$

3. Find an equation of the tangent line to the
(3) graph of $y = g(x)$ at $x = 5$, if $g(5) = -3$ and $g'(5) = 4$.

Solution: Let $y = g(x)$. Given: $g(5) = -3$, $g'(5) = 4$.

$$(\text{slope})\ m = y' = g'(x) = g'(x)/_{x=5} = 4$$
$$y = g(x) = g(x)/_{x=5} = -3.$$

$x_1 = 5$, $y_1 = -3$ and $m = 4$.

$\therefore$ The equation of the tangent line is,

$$y - y_1 = m(x - x_1)$$
$$y + 3 = 4(x - 5).$$
$$y = 4x - 23.$$

4. If an equation of the tangent line to the curve
(4) $y = f(x)$ at point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

Solution: Given: $y = f(x) = 4x - 5$

$$f'(x) = 4$$
$$f(2) = 8 - 5 = 3.$$
$$m = f'(x)/_{x=2} = f'(2) = 4$$
$$\therefore f(2) = 3,\ f'(2) = 4.$$

5. Find the tangent line to the equation $x^3+y^3=6xy$ at the point $(3,3)$ and at what point the tangent line horizontal in the first Quadrant. [U.Q : 2018 Jan]

(6)

Solution: Given: $x^3+y^3=6xy$ at $(3,3)$.

$$\left[\Rightarrow 6xy = x^3+y^3 \right.$$

$$\left.\Rightarrow y = \frac{x^3+y^3}{6x} = \frac{x^3}{6x}+\frac{y^3}{6x}\right] \times$$

$$x^3+y^3=6xy \rightarrow (1)$$

$$3x^2+3y^2y' = 6(1.y+xy')$$

$$3x^2+3y^2y' = 6y+6xy'$$

$$3y^2y'-6xy' = 6y-3x^2$$

$$y' = \frac{6y-3x^2}{3y^2-6x} \quad \text{at } 3,3.$$

$$y' = -1 \Rightarrow \frac{dy}{dx} = -1. \Rightarrow m=-1$$

The equation of the tangent line at $(3,3)$ is,

$$y-y_1 = m(x-x_1)$$

$$y-3 = (-1)(x-3)$$

$$y-3 = -x+3$$

$$\Rightarrow x+y = 6.$$

To find the point of the tangent line horizontal in the first quadrant.

$$y' = 0$$

$$\therefore y' = \frac{6y-3x^2}{3y^2-6x} = 0$$

$$\Rightarrow 6y-3x^2=0 \Rightarrow 2y = x^2$$

$$\therefore y = x^2/2.$$

① ⇒ $x^3 + y^3 = 6xy$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$\Rightarrow x^3 + \frac{x^6}{8} = 3x^3 \quad \Rightarrow \frac{x^6}{8} = 2x^3$$

$$\Rightarrow x^3 = 16 \quad \Rightarrow x^3 = (2)^4$$

$$\therefore x = 2^{4/3}$$

$x = 2^{4/3}$ put in $y = x^2/2$

$$\Rightarrow y = \frac{(2^{4/3})^2}{2} = 2^{8/3 - 1}$$

$$y = 2^{5/3}.$$

∴ The points are $x = 2^{4/3}$, $y = 2^{5/3}$.

6. Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Solution: $y = -2x$.

7. Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.

Find equations of the tangent lines at the points $(1, 1)$ and $(4, 1/2)$.

1.5 Differentiation Rules:

a) Derivatives of polynomials:

1) Equation of tangent line is $y - y_1 = m(x - x_1)$, $m = \frac{dy}{dx}$.

2) Equation of normal line is $y - y_1 = -\frac{1}{m}(x - x_1)$

1. Find an equation of the tangent line and normal line to the curve the given point $y = 3x^2 - x^3$, $(1, 2)$.

6

Solution: Given : $y = 3x^2 - x^3$, $(1,2)$

$$y = 3x^2 - x^3 \Rightarrow y' = 6x - 3x^2$$

$$m = (y')_{1,2} = 6 - 3 = 3.$$

a) Equation of tangent line is,

$$y - y_1 = m(x - x_1)$$
$$(y - 2) = 3(x - 1)$$
$$y - 2 = 3x - 3$$
$$\Rightarrow y = 3x - 1$$

b) Equation of normal line is,

$$y - y_1 = -\frac{1}{m}(x - x_1)$$
$$(y - 2) = -\frac{1}{3}(x - 1)$$
$$3y - 6 = -x + 1$$
$$3y = -x + 7$$
$$y = -\frac{1}{3}x + \frac{7}{3}.$$

2. H.W $y = \sqrt[4]{x} = (x)^{1/4}$, Ans: $y = \frac{1}{4}x + \frac{3}{4}$, $y = -4x + 5$.

3. The equation of motion of a particle is $S = 2t^3 - 5t^2 + 3t + 7$, where S is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Solution: The velocity and acceleration are,

$$V(t) = \frac{ds}{dt} = 6t^2 - 10t + 3.$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

$$[a(t)]_{t=2} = 14 \text{ cm}/s^2$$

4. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

a) The velocity and acceleration as function of t,

b) The acceleration after 2 s and

c) the acceleration when the velocity is 0.

Solution: Given: $s = t^3 - 3t$.

a) $$v = \frac{ds}{dt} = 3t^2 - 3$$

$$a = \frac{dv}{dt} = 6t$$

b) If acceleration after 2 seconds, then

$$\left(\frac{dv}{dt}\right)_{t=2} = 12 \text{ m}/s^2$$

c) Find acceleration when the velocity is 0.

$$v = 0 \rightarrow 3t^2 - 3 = 0 \Rightarrow 3t^2 = 3$$

$$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1.$$

$$\therefore \left(\frac{dv}{dt}\right)_{t=1} = 6 \text{ m}/s^2 \quad [\text{reject } t = -1].$$

Derivatives:

5. Find the first and second derivatives of the function $f(x) = 10x^{10} + 5x^5 - x$.

Solution: Given: $f(x) = 10x^{10} + 5x^5 - x$

$$f'(x) = 10\left[10x^9\right] + 5\left[5x^4\right] - 1$$

$$= 100x^9 + 25x^4 - 1.$$

$$f''(x) = 100\left[9x^8\right] + 25\left[4x^3\right] - 0$$
$$= 900x^8 + 100x^3.$$

6. Find y'' if $x^4 + y^4 = 16$.

Solution: Given: $x^4 + y^4 = 16$.

$$\Rightarrow y^4 = -x^4 + 16.$$
$$\Rightarrow 4y^3y' = -4x^3 + 0$$
$$\Rightarrow y' = -x^3/y^3$$

$$y'' = -\left[\frac{y^3 3x^2 - x^3 3y^2 y'}{(y^3)^2}\right] = -\left[\frac{3x^2y^3 - 3y^2x^3\left(-x^3/y^3\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 + 3x^6/y}{y^6}\right] = -\left[\frac{3x^2y^3}{y^6} + \frac{3x^6}{y^7}\right] = \left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$$

$$= -\left[\frac{3x^2(-x^4+16) + 3x^6}{y^7}\right] = -\left[\frac{-3x^6 + 48x^2 + 3x^6}{y^7}\right]$$

$$y'' = \frac{-48x^2}{y^7}.$$

7. Find a second degree polynomial p such that $p(2) = 5$, $p'(2) = 3$, $p''(2) = 2$.

Solution: Given: $p(2) = 5$, $p'(2) = 3$, $p''(2) = 2$.

Let $p(x) = ax^2 + bx + c \rightarrow (1)$

$p'(x) = 2ax + b \rightarrow (2)$

$p''(x) = 2a \rightarrow (3)$

Given: $p''(2) = 2 \Rightarrow p''(2) = 2a = 2 \Rightarrow a = 1$

$p'(2) = 3 \Rightarrow p'(2) = 4a + b = 3$

$\Rightarrow 4 + b = 3$

$b = -1$

$p(2) = 5 \Rightarrow p(2) = 4a + 2b + c = 5 \Rightarrow 4 - 2 + c = 5 \Rightarrow c = 3.$

$\therefore (1) \Rightarrow p(x) = x^2 - x + 3$

1.6 Maxima and Minima of functions of one variable: (5)

a) Absolute Maximum and absolute Minimum:

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the

* absolute Maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
* absolute Minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition: The number $f(c)$ is a

* local Maximum value of f if $f(c) \geq f(x)$ when x is near c.
* local Minimum value of f if $f(c) \leq f(x)$ when x is near c.

Extreme value Theorem:

If f is continuous on a closed interval $[a,b]$, then f attains an absolute Maximum value $f(c)$ and an absolute Minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Fermat's Theorem:

If f has a local Maximum or Minimum at c and if $f'(c)$ exists then $f'(c)=0$.

Definition:

A Critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method:

To find the absolute Maximum and Minimum values of a continuous function f on a closed interval $[a, b]$.

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the end points of the interval.
3. The largest of the values from steps 1 and 2 is the absolute Maximum value; the smallest of these values is the absolute Minimum value.

1. Find the critical values of the function:

$f(x) = 5x^2+4x.$

Solution: Given: $f(x) = 5x^2+4x \rightarrow (1)$

Critical numbers of f occur at $f'(x)=0 \rightarrow (2)$

$$f'(x) = 10x+4 \rightarrow (3)$$

$(2) \Rightarrow \quad 10x+4=0 \Rightarrow x = -4/10 \Rightarrow x = -2/5$

$\therefore$ The critical value $= -2/5$.

2. $g(x) = 2x^3-3x^2-36x.$

Solution: Given: $g(x) = 2x^3-3x^2-36x. \rightarrow (1)$

critical numbers of g occur at $g'(x)=0$. $\rightarrow$ (2) (16)

$\Rightarrow g'(x) = 6x^2 - 6x - 36$

(2) $\Rightarrow$ $6x^2 - 6x - 36 = 0$

$\Rightarrow x = -2, x = 3.$

$\therefore$ Critical value $= -2, 3.$

3. $f(x) = x^2 e^{-3x}$. Ans: $x=0$, $x = 2/3$.

4. $f(x) = x^2 + 2/x$ Ans: $x = 1$.

5. Find the absolute and local Maximum and minimum values of

$f(x) = \frac{1}{x}$, $x \geq 1$.

Solution: Given: $f(x) = \frac{1}{x}$, $x \geq 1$.

x	1	2	3	....
$f(x)$	1	$1/2$	$1/3$	....

Maximum

$f(1) = \frac{1}{1} = 1$ is the absolute Minimum of f.

$f(x) = 1/x = 1/1 = 1$

6. $f(x) = \begin{cases} 1-x & \text{if } 0 \leq x < 2 \\ 2x-4 & \text{if } 2 \leq x \leq 3. \end{cases}$ — ?

$c=1$ $f(c) = 1$

$f(c) = f(x)$.

Solution:

x	0	1	2	3
$f(x)$	1	0	0	2

$f(3) = 6-4 = 2$ is the absolute Maximum.

$= f(0) = 1-x = 1-0 = 1$ is a Min. $c=2$.

7. $f(x) = |x|$, $-1 < x < 2$.

Ans: $f(0) = 0$ is the A. Minimum.

No, absolute Maximum.

The closed interval Method.

1. Find the absolute Maximum and Minimum values of $f(x) = x^2$, $[-2,1]$

Solution: Given: $f(x) = x^2$, $[-2,1]$ $\to$ ①

Critical numbers of f occur at $f'(x)=0$ $\to$ ②

$$f'(x) = 2x$$

② $\Rightarrow$ $2x = 0$ $\Rightarrow x = 0.$

$\therefore$ Critical value $= 0.$

$f(-2) = 4$ is the a. Maximum value of f

$f(0) = 0$ is the a. Minimum value of f.

	end point	Critical point	end point
x	-2	0	1
$f(x)$	4	0	1

2. $f(x) = 3x^4 - 16x^3 + 18x^2$, $-1 \le x \le 4$

Solution: Given: $f(x) = 3x^4 - 16x^3 + 18x^2$. $-1 \le x \le 4$ $\to$ ①

Critical numbers of f occur at $f'(x) = 0$ $\to$ ②

$$f'(x) = 12x^3 - 48x^2 + 36x$$

② $\Rightarrow$ $12x^3 - 48x^2 + 36x = 0$

$\Rightarrow 12x\left[x^2 - 4x + 3\right] = 0$

$\Rightarrow x = 0, 3, 1.$

$\therefore$ Critical value $= 0, 1, 3.$

	end point	Critical point	Critical point	Critical point	End point
x	-1	0	1	3	4
$f(x)$	a.M 37	0	5	-27	32

$f(3) = -27$ is the a. Minimum value of f

$f(-1) = 37$ is the a. Maximum value of f

3. H.W. $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x \le 4$. [Ans: C.V $= 0, 2$ $f(4) = 17$, $f(2) = -3$]

1.6 (b) Increasing / Decreasing test:

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

First Derivative Test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local Maximum at c.

b) If f' changes from negative to positive at c, then f has a local Minimum at c.

c) If f' does not change sign at c (ex: if f' is positive on both sides of c or negative on both sides) then f has no local Maximum or Minimum at c.

Concave upward / concave downward:

If the graph of f lies above all of its tangents on an interval I, then it is called Concave upward on I, If the graph of f lies between all of its tangents on I, it is called Concave downward on I.

Concave upward ≡ Concave downward.

Concavity Test:

a) If $f''(x) > 0$ for all x in I, then the graph of f is Concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is Concave downward on I.

Inflection point:

A point P on a curve $y=f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Second Derivative Test:

Suppose f'' is continuous near c.

a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.

b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Answer the following questions about the functions whose derivatives are given:

a) What are the critical point of f?

b) On what interval is f increasing or decreasing?

c) At what points, if any, does f assume local maximum and minimum values?

d) Find intervals of concavity and the inflection points.

1. $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ [U.Q Jan 2016]

Solution:

Given: $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$.

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\pi/4$, $5\pi/4$. $\sin x = \cos x$. $1/\sqrt{2}$.

$\sin(180+45)$

$-\sin 45°$

$\sin(540+\theta) = -\cos\theta$

$\sin[270+\theta] = -\cos\theta$

a) critical points are $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\pi/4 + 360° = 2\pi + \pi/4 = 7\pi/4$

b)

Interval	Sign of f'	Behaviour of f
$0 < x < \pi/4$	+	increasing
$\pi/4 < x < 5\pi/4$	−	decreasing
$5\pi/4 < x < 2\pi$	+	increasing.

c) The first derivative test is a local.

(i) Maximum at $\pi/4$, $f(\pi/4) = \sin\pi/4 + \cos\pi/4$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}.$$

(ii) Minimum at $5\pi/4$, $f(5\pi/4) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$

$$= -\sqrt{2}$$

d) $f'(x) = -\sin x - \cos x = -(\sin x + \cos x)$

$f'(x) = 0 \Rightarrow -(\sin x + \cos x) = 0$

$\Rightarrow \sin x = -\cos x$

$\cos(90+45)$ $(270+\theta)$

$= -\sin 45 = \frac{\pi}{2} + \pi/4 = 3\pi/4$ $\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$. $\sin(270+45)$

$= -\cos 45 = \frac{3\pi}{2} + \pi/4 = \frac{7\pi}{4}$

$\pi - \pi/4$

$\pi/4 + \pi$

Interval	Sign of f''	Behaviour of f
$0 < x < 3\pi/4$	−	Concave down
$\frac{3\pi}{4} < x < 7\pi/4$	+	Concave up
$7\pi/4 < x < 2\pi$	−	concave down

e) Inflection points are,

$$\left(\frac{3\pi}{4}, 0\right)\left(\frac{7\pi}{4}, 0\right)$$

$\left[\because \text{since, } f\left(\frac{3\pi}{4}\right) = 0,\ f\left(\frac{7\pi}{4}\right) = 0.\right]$

2. $f(x) = x^4 - 2x^2 + 3$ [U.Q Jan 2016]

Solution : Given: $f(x) = x^4 - 2x^2 + 3$

$\Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2-1)$

$= 4x(x-1)(x+1)$

$f'(x) = 0 \Rightarrow 4x(x-1)(x+1) = 0$

$\Rightarrow x = 0, x = 1, x = -1.$

a) Critical points are $x = 0, x = 1, x = -1.$

b).

Interval	Sign of f'	Behavior of f.
$-\infty < x < -1$	$-$	decreasing
$-1 < x < 0$	$+$	increasing
$0 < x < 1$	$-$	decreasing
$1 < x < \infty$	$+$	increasing

local min at $x=-1$

local max at $x=0$

local min at $x=1$

c) The first derivative test is a local,

(i) a local Maximum at $x = 0$, $f(0) = 3$.

(ii) a local Minimum at $x = \pm 1$, $f(\pm 1) = 2$.

d) $f''(x) = 12x^2 - 4$

$f''(x) = 0 \Rightarrow 12x^2 - 4 = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow 3x^2 = 1$

$\Rightarrow x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3}$.

Interval	Sign of f''	Behaviour of f
$-\infty < x < -1/\sqrt{3}$	$+$	Concave up
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$-$	Concave down
$\frac{1}{\sqrt{3}} < x < \infty$	$+$	Concave up

e) Inflection points are $\left(\pm\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$

since, $f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{22}{9}$.

H.W.

3. $f(x) = x + 2\sin x,\ 0 \le x \le 2\pi$

4. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ $\qquad 3x^4 - 16x^3 + 18x^2,\ -1 \le x \le 4$.

5. Find the local Maximum and Minimum values of f using both the first and second derivative tests. $y = x^4 - 4x^3$ with respect to concavity and points of inflection.

Solution: Given: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0$

$\Rightarrow 4x^2(x-3) = 0$

$\Rightarrow x = 0,\ x = 3.$ $\qquad -\infty\ (-)\ 0\ (-)\ 3\ +\ \infty$ min.

a) $\therefore$ The critical values are $0, 3$.

b)

Interval	Sign of f'	Behaviour of f
$(-\infty, 0)$	$-$	decreasing
$(0, 3)$	$-$	decreasing
$(3, \infty)$	$+$	Increasing

c) First derivative test tells us that f does not have a local Maximum or Minimum at 0.

d) $f''(x) = 12x^2 - 24x$

$f''(x) = 0 \Rightarrow 12x^2 - 24x = 0$

$\Rightarrow 12x(x-2) = 0$

$\Rightarrow x = 0,\ x = 2.$

Interval	$f''(x)$	Behaviour of f
$(-\infty, 0)$	+	Concave up
$(0, 2)$	−	Concave down
$(2, \infty)$	+	Concave up

e) Inflection points are,

$$(0,0) \quad (2,-16)$$

$\because$ Since, $f(0) = 0$, $f(2) = -16$.

f) The second derivative test:

$$f''(c) = 0.$$

$$f''(x) = 12x^2 - 24x.$$

$f'(3) = 0$, $f''(3) > 0$, $f(3) = -27$ is a local minimum.

The second derivative test gives no information about the critical number 0.

Since $f'(0) = 0$, $f''(0) = 0$.

But first derivative test gives f does not have a local maximum or minimum at 0.

H.W

6. Find the local maximum and minimum values of $f(x) = x^5 - 5x + 3$ using both the first and second derivative tests.

Ans: $f'(x) = 5x^4 - 5$ $\quad f'(1) = 0$, $f'(-1) = 0$. $f''(1) = 20$, $f''(-1) = -20$.

$x = 1, -1$.

1^{st} derivative:

i) Max $f(-1) = 7$

ii) Min $f(1) = -1$

2^{nd} derivative:

i) Max $f(-1) = 7$

ii) Min $f(1) = -1$.

① For the function $f(x) = 2x^3 + 3x^2 - 36x$

(i) Find the intervals on which it is increasing or decreasing.

(ii) Find the local max & local min. values of f.

(iii) Find the intervals of concavity & the inflection points.

$\longrightarrow$ (i) Given $f(x) = 2x^3 + 3x^2 - 36x$

$$f'(x) = 6x^2 + 6x - 36$$

$$f'(x) = 0 \Rightarrow 6(x^2 + x - 6) = 0 \Rightarrow 6(x+3)(x-2) = 0.$$

$$\Rightarrow x = -3, 2.$$

$+ \quad\quad - \quad\quad +$

$-\infty \quad\quad -3 \quad\quad 2 \quad\quad \infty$

Take $x = -4$: $f'(x) = 6(-4+3)(-4-2) = 6(-1)(-6) = 36$

$x = 0$: $f'(x) = 6(3)(-2) = -36$

$x = 3$: $f'(x) = 6(6)(1) = 36$

Interval	Nature of $f'(x)$	f
$(-\infty, -3)$	$+$	$\uparrow$
$(-3, 2)$	$-$	$\downarrow$
$(2, \infty)$	$+$	$\uparrow$

$\Rightarrow f$ is max. at $x = -3$

$\Rightarrow f$ is min at $x = 2$.

(ii) $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = -54 + 27 + 108 = 81$

$f(2) = 2(2)^3 + 3(2)^2 - 36(2) = 16 + 12 - 72 = -44$

(iii) $f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

$f''(x) = 0 \Rightarrow 12x + 6 = 0$

$\Rightarrow x = \frac{-6}{12} = \frac{-1}{2}.$

At $x = -\frac{1}{2}$,

$f(x) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 36\left(-\frac{1}{2}\right)$

$= -\frac{37}{2}.$

$-$ $+$

$-\infty$ $-\frac{1}{2}$ ∞

Take

$x = -1$	$x = 0$
$f''(x) = 12(-1) + 6 = -6$	$f''(x) = 6$

Interval	Nature of f''	f
$(-\infty, -\frac{1}{2})$	$-$	Concave downward
$(-\frac{1}{2}, \infty)$	$+$	Concave upward

Here inflection point is $\left(-\frac{1}{2}, -\frac{37}{2}\right)$.

(2) $x^4 - 4x^3$

(3) $3x^4 - 4x^3 - 12x^2 - 5$

(i) $\longrightarrow$ $f(x) = 3x^4 - 4x^3 - 12x^2 - 5.$

$f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0.$

$\Rightarrow 12x(x^2 - x - 2) = 0.$

$\Rightarrow 12x(x-2)(x+1) = 0$ ✓

$\Rightarrow x = 0, 2, -1.$

$\Rightarrow x = -1, 0, 2$ (critical Points)

$-$ $+$ $-$ $+$

$-\infty$ (-1 x 0 x 2 x ∞)

Take $x=-2$	$x=-\frac{1}{2}$	$x=1$	$x=3$
$f'(x) = 12(-2)(-2-2)(-2+1)$	$f' = [12(-\frac{1}{2})(-\frac{1}{2}-2) \cdot (-\frac{1}{2}+1)]$	$f' = 12(1)(1-2)(1+1)$	$f' = 12(3)(3-2)(3+1)$
$= 12(-2)(-4)(-1)$	$= 12(-\frac{1}{2})(-\frac{5}{2})(\frac{1}{2})$	$= 12(1)(-1)(2)$	$= 12(3)(+1)(4)$
$= -96$	$= +\frac{15}{2}$	$= -24$	$= +144$

Interval	f'	f
$(-\infty, -1)$	$-$	Decreasing
$(-1, 0)$	$+$	Incresing
$(0, 2)$	$-$	Decreasing
$(2, \infty)$	$+$	Incresing.

$f''(x) = 0 \Rightarrow 12x^2 - 24x = 0$.

$\Rightarrow 12x(x-2) = 0$ ✓

$\Rightarrow x = 0, 2$

$$\left[\begin{array}{l} 12x(x-2) = 0. \\ \Rightarrow 12 \neq 0,\ x = 0,\ x-2 = 0 \end{array}\right.$$

$$\begin{array}{ccccc} & + & & - & & + & \\ -\infty & & 0 & & 2 & & \infty \end{array}$$

$(-\infty, 0)\ (0, 2)\ (2, \infty)$

Take

$x = -1$	$x = 1$	$x = 3$
$f''(x) = 12(-1)(-1-2)$	$f'' = 12(1)(1-2)$	$f'' = 12(3)(3-2)$
$= 12(-1)(-3)$	$= 12(1)(-1)$	$= 12(3)(1)$
$= +36$	$= -12$	$= +36$

Interval	f''	f
$(-\infty, 0)$	$+$	Concave upwards
$(0, 2)$	$-$	Concave downwards
$(2, \infty)$	$+$	Concave upwards

$\boxed{\text{Try}}$ (3) $f(x) = 3x^4 - 4x^3 - 12x^2 - 5$

$\rightarrow$ or move to next page.

PTO

(ii) $f'(x) = 12x^3 - 12x^2 - 24x$

$f''(x) = 36x^2 - 24x - 24$

$f''(x) = 0 \Rightarrow 36x^2 - 24x - 24 = 0.$

$\Rightarrow 12(3x^2 - 2x - 2) = 0.$ ✓

-6

-2

$3x^2 - 2x - 2 = 0.$

$\Rightarrow \quad x = \dfrac{2 \pm \sqrt{4+24}}{6}$

$= \dfrac{2 \pm \sqrt{28}}{6} = \dfrac{2 \pm \sqrt{4\times7}}{6} = \dfrac{2 \pm 2\sqrt{7}}{6} = \dfrac{1 \pm \sqrt{7}}{3}.$

$\simeq \dfrac{1 \pm 2.6}{3} \simeq \dfrac{3.6}{3}, \dfrac{-1.6}{3}$

$\simeq 1.2, \ -0.5$

$+ \qquad - \qquad +$

$-\infty \qquad -0.5 \qquad 1.2 \qquad \infty$

$(\qquad \times \qquad \times \qquad)$

Take

$x = -1$	$x = 0$	$x = 2$
$f'' = 12(3+2-2)$	$f'' = 12(-2)$	$f'' = 12(12-4-2)$
$= +36$	$= -24$	$= 12(6)$
		$= +72$

Interval	f''	f
$(-\infty, -0.5)$	$+$	Increasing
$(-0.5, 1.2)$	$-$	Decreasing
$(1.2, \infty)$	$+$	Increasing