LAPLACE INTEGRATION AND ITS APPLICATION

DR. P.AGILAN M.Sc, B.Ed, M.Phil, Ph.D
ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS
ST.JOSEPH'S COLLEGE OF ENGINEERING, OMR, CHENNAI-119

LAPLACE TRANSFORM DEFINITION

Definition. Let $f(t)$ be defined for all $t \geq 0$, then $\int_0^{\infty} e^{-st} f(t)dt$ is defined as the Laplace transform of $f(t)$, if the integral exists.

It is denoted by $L[f(t)]$. It is a function of s.

$$\text{Hence, } L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = F(s).$$

Result. The parameter s may be real or complex.

SUFFICIENT CONDITION

Sufficient condition for existence of Laplace Transforms

[May2015, Dec 2014, May 2011]

Let $f(t)$ be defined for all $t \geq 0$ such that (i) $f(t)$ is piecewise continuous in the interval $[0, \infty)$ and (ii) $f(t)$ is of exponential order $\alpha > 0$, then the Laplace transform of $f(t)$ exists for $s > \alpha$.

Note

1. Piecewise continuous on $[0, \infty)$ means that the function is continuous on every finite subinterval $0 \leq t \leq \alpha$ except possibly at a finite number of points where they have jumps i.e., $f(x+)$ and $f(x-)$ exist but not equal.
2. $f(t)$ is of exponential order $\alpha > 0$ if $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$ and M is a constant. In other words, $\lim_{t \to \infty} (e^{-\alpha t} f(t))$ is finite.

EXAMPLE

Example. t^n is of exponential order as $t \to \infty$.

$$\lim_{t\to\infty} (e^{-\alpha t} t^n) = \lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} = \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} = 0.$$

Result. The above conditions are sufficient but not necessary.

Example. $L\left[\frac{1}{\sqrt{t}}\right]$ exists but it is not continuous at $t = 0$.

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Laplace Transform of Standard functions

1. $L[1] = \int_0^\infty e^{-st} 1 dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}.$

2. $L[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}[0-1] = \frac{1}{s-a}.$

3. $L[e^{-at}] = \frac{1}{s+a}.$

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

4. $L[t]=\int_0^\infty e^{-st}t\,dt=\int_0^\infty t\,d\left(\frac{e^{-st}}{-s}\right)=\left[t\frac{e^{-st}}{-s}\right]_0^\infty-\int_0^\infty\frac{e^{-st}}{-s}dt=\frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty=\frac{1}{s^2}.$

5. $L[t^n]=\int_0^\infty e^{-st}t^n dt=\int_0^\infty t^n d\left(\frac{e^{-st}}{-s}\right)=\left[t^n\frac{e^{-st}}{-s}\right]_0^\infty-\int_0^\infty\frac{e^{-st}}{-s}nt^{n-1}dt$

$$=\frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt=\frac{n}{s}L[t^{n-1}]=\frac{n}{s}\frac{n-1}{s}L[t^{n-2}]$$

$$=\frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots L[t]$$

$$=\frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{1}{s^2}=\frac{n!}{s^{n+1}}.$$

6. $L[\cosh at]=L\left[\frac{e^{at}+e^{-at}}{2}\right]=\frac{1}{2}\left[L[e^{at}]+L[e^{-at}]\right]=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]=\frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]=\frac{s}{s^2-a^2}.$$

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

7. $L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}.$$

8. $L[\cos at] = L\left[\frac{e^{iat} + e^{-iat}}{2}\right] = \frac{1}{2}\left[\frac{1}{s-ai} + \frac{1}{s+ai}\right]$

$$= \frac{1}{2}\left[\frac{s+ai+s-ai}{s^2+a^2}\right] = \frac{s}{s^2+a^2}.$$

9. $L[\sin at] = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right] = \frac{1}{2i}\left[\frac{1}{s-ai} - \frac{1}{s+ai}\right]$

$$= \frac{1}{2i}\left[\frac{s+ai-s+ai}{s^2+a^2}\right] = \frac{1}{2i}\left[\frac{2ai}{s^2+a^2}\right] = \frac{a}{s^2+a^2}.$$

EXAMPLE

Example Write a function for which Laplace transform does not exist. Explain why Laplace transform does not exist? [May 2007]

Solution. Let $f(t) = e^{t^2}$.

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} e^{t^2}$$

$$= \lim_{t\to\infty} e^{-st+t^2}$$

$$= \lim_{t\to\infty} e^{t(-s+t)} = e^{\infty} = \infty.$$

$\therefore \lim_{t\to\infty} e^{-st} e^{t^2}$ is not finite.

i.e., e^{t^2} is not of exponential order.

$\therefore$ Laplace transform of $f(t)$ does not exist.

LINEARITY PROPERTY

Linearity Property

If $f(t)$ and $g(t)$ are two continuous functions of t and k is a constant, then

(i) $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$.

(ii) $L[kf(t)] = kL[f(t)]$ where $k \neq 0$.

EXAMPLE

Example Is the linearity property applicable to $L\left[\frac{1-\cos t}{t}\right]$? Reason out. [Dec 2012]

Solution. Linearity property can be applied when $\frac{1}{t}$ and $\frac{\cos t}{t}$ are continuous. Here, $\frac{1}{t}$ and $\frac{\cos t}{t}$ are not even defined at $t = 0$. Hence, they are not continuous. Therefore, linearity property is not applicable to $\frac{1-\cos t}{t}$.

Example Find $L[3e^{5t} + 5\cos t]$. [Jan 2009]

Solution. $L[3e^{5t} + 5\cos t] = 3L[e^{5t}] + 5L[cost]$

$$= 3\frac{1}{s-5} + 5\frac{s}{s^2+1} = \frac{3}{s-5} + \frac{5s}{s^2+1}.$$

Example Find $L[\cos \pi t + e^{-\frac{2}{3}t} + \sin 8t]$. [Jun 2009]

Solution. $L[\cos \pi t + e^{-\frac{2}{3}t} + \sin 8t] = L[\cos \pi t] + L[e^{-\frac{2}{3}t}] + L[\sin 8t]$

$$= \frac{s}{s^2+\pi^2} + \frac{1}{s+\frac{2}{3}} + \frac{8}{s^2+64}.$$

EXAMPLE

Example Find $L[t^2 + e^{-5t} + 8 + \sinh 5t]$. [Jun 2007]

Solution. $L[t^2 + e^{-5t} + 8 + \sinh 5t] = L[t^2] + L[e^{-5t}] + 8L[1] + L[\sinh 5t]$

$$= \frac{2!}{s^3} + \frac{1}{s+5} + 8\frac{1}{s} + \frac{5}{s^2 - 25} = \frac{2}{s^3} + \frac{1}{s+5} + \frac{8}{s} + \frac{5}{s^2 - 25}.$$

Example Find $L[(t+1)^2]$. [Jun 2002]

Solution. $L[(t+1)^2] = L[t^2 + 2t + 1]$

$$= L[t^2] + 2L[t] + L[1] = \frac{2!}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} = \frac{2 + 2s + s^2}{s^3}.$$

Example Find $L[\cos^2 2t]$. [Dec 2005]

Solution. $L[\cos^2 2t] = L\left[\frac{1 + \cos 4t}{2}\right] = \frac{1}{2}\left[L[1] + L[\cos 4t]\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 16}\right]$

$$= \frac{1}{2}\left[\frac{s^2 + 16 + s^2}{s(s^2 + 16)}\right] = \frac{1}{2}\left[\frac{2s^2 + 16}{s(s^2 + 16)}\right] = \frac{s^2 + 8}{s(s^2 + 16)}.$$

EXAMPLE

Example Find $L[\sin 2t \sin 3t]$. [Dec 2004]

Solution. $L[\sin 2t \sin 3t] = \frac{1}{2}L[\cos t - \cos 5t] = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$

$$= \frac{s}{2}\left[\frac{s^2+25-s^2-1}{(s^2+1)(s^2+25)}\right] = \frac{s}{2}\left[\frac{24}{(s^2+1)(s^2+25)}\right]$$

$$= \frac{12s}{(s^2+1)(s^2+25)}.$$

Example Find $L[\sin^3 2t]$. [May 2004]

Solution. $L[\sin^3 2t] = L\left[\frac{1}{4}(3\sin 2t - \sin 6t)\right] = \frac{1}{4}\left[3\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$

$$= \frac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right] = \frac{3}{2}\left[\frac{32}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}.$$

EXAMPLE

Example Find $L[t^{\frac{-1}{2}}]$. [May 2005]

Solution. $L[t^{-\frac{1}{2}}] = \int_0^\infty e^{-st} t^{-\frac{1}{2}} dt$

Let $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{1}{s} dx$.

When $t = 0, x = 0$, when $t = \infty, x = \infty$

$$\therefore L[t^{\frac{-1}{2}}] = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^{-\frac{1}{2}} \frac{dt}{s} = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{\frac{1}{2}-1} dx$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) \qquad \left[\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx\right]$$

$$= \frac{1}{\sqrt{s}} \sqrt{\pi} = \sqrt{\frac{\pi}{s}}.$$

EXAMPLE

Example Find $L[\cos 4t.\sin 2t]$. [Dec 2009]

Solution. $L[\cos 4t.\sin 2t] = L\left[\frac{\sin 6t - \sin 2t}{2}\right] = \frac{1}{2}[L[\sin 6t] - L[\sin 2t]]$

$$= \frac{1}{2}\left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] = \frac{3}{s^2+36} - \frac{1}{s^2+4}.$$

Example Find $L[\sin^2 t \cos^3 t]$. [May 2008]

Solution. We have $\sin^2 t \cos^3 t = \left(\frac{1-\cos 2t}{2}\right)\left(\frac{\cos 3t + 3\cos t}{4}\right)$

$$= \frac{1}{8}[\cos 3t + 3\cos t - \cos 2t \cos 3t - 3\cos 2t \cos t]$$

$$= \frac{1}{8}\left[\cos 3t + 3\cos t - \frac{1}{2}(\cos 5t + \cos t) - \frac{3}{2}(\cos 3t + \cos t)\right]$$

$$= \frac{1}{8}\left[\cos 3t + 3\cos t - \frac{1}{2}\cos 5t - \frac{1}{2}\cos t - \frac{3}{2}\cos 3t - \frac{3}{2}\cos t\right]$$

$$= \frac{1}{8}\left[-\frac{1}{2}\cos 3t + \cos t - \frac{1}{2}\cos 5t\right]$$

$$L[\sin^2 t \cos^3 t] = L\left[\frac{1}{8}\left[-\frac{1}{2}\cos 3t + \cos t - \frac{1}{2}\cos 5t\right]\right]$$

$$= \frac{1}{16}L[2\cos t - \cos 3t - \cos 5t]$$

$$= \frac{1}{16}\left[\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right].$$

EXAMPLE

Example Find the Laplace transform of the function defined by

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}.$$

[Dec 2007]

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} \cdot tdt + \int_1^\infty e^{-st} \times 0dt$$

$$= \int_0^1 e^{-st} tdt = \int_0^1 td\left(\frac{e^{-st}}{-s}\right)$$

$$= \left[t\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}dt.$$

$$= -\frac{1}{s}[e^{-s} - 0] + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1)$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}.$$

EXAMPLE

Example Find the Laplace transform of $f(t)$ defined by $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1. \end{cases}$

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} e^t dt = \int_0^1 e^{-(s-1)t} dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = -\frac{1}{s-1}\left[e^{-(s-1)} - 1\right] = \frac{1 - e^{-(s-1)}}{s-1}.$$

EXAMPLE

Example Find the Laplace transform of the function defined by

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}.$$

[May 2006]

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt$$

$$= \int_0^\pi e^{-st} \sin t dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi - \cos\pi) - \frac{1}{s^2+1}(-s\sin 0 - \cos 0) = \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}.$$

PROPERTIES OF LALACE TRANSFORMS

Shifting Property - First Shifting theorem

1. **Statement.** If $L[f(t)] = F(s)$ then $L[e^{-at} f(t)] = F(s + a)$.

Proof. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)dt = \int_0^\infty e^{-(st+at)} f(t)dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)dt = F(s + a) = [L[f(t)]]_{s \to s+a}$$

2. $L[e^{at} f(t)] = F(s - a) = L[f(t)]_{s \to s-a}$.

PROPERTIES OF LALACE TRANSFORMS

Change of scale property

Statement. If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right), a > 0.$

Proof. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

Let $at = x \quad \Rightarrow t = \frac{x}{a} \Rightarrow dt = \frac{1}{a}dx$

When $t = 0, x = 0,$ when $t = \infty, x = \infty$

$$\therefore L[f(at)] = \int_0^\infty e^{-s\frac{x}{a}} f(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}[F(s)]_{s\to\frac{s}{a}} = \frac{1}{a}[L[f(t)]]_{s\to\frac{s}{a}}.$$

PROBLEMS

Example Find $L[e^t t^{-\frac{1}{2}}]$. [May 1996]

Solution. $L[e^t t^{-\frac{1}{2}}] = L\left[t^{-\frac{1}{2}}\right]_{s\to s-1} = \left[\sqrt{\frac{\pi}{s}}\right]_{s\to s-1} = \sqrt{\frac{\pi}{s-1}}.$

Example Find the Laplace transform of $\frac{t}{e^t}$. [Jun 2013]

Solution. $L\left[\frac{t}{e^t}\right] = L[e^{-t}t] = L[t]_{s\to s+1} = \left[\frac{1}{s^2}\right]_{s\to s+1} = \frac{1}{(s+1)^2}.$

Example Find $L\left[e^{-3t}\sin t \cdot \cos t\right]$ [Dec 2011]

Solution. $L\left[e^{-3t}\sin t\cos t\right] = L\left[e^{-3t}\frac{\sin 2t}{2}\right] = \frac{1}{2}L\left[e^{-3t}\sin 2t\right]$

$$= \frac{1}{2}L[\sin 2t]_{s\to s+3} = \frac{1}{2}\left(\frac{2}{s^2+4}\right)_{s\to s+3}$$

$$= \frac{1}{(s+3)^2+4} = \frac{1}{s^2+6s+13}.$$

Example Find $L[e^{-3t}\sin^2 t]$. [Jun 2008]

Solution. $L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s\to s+3} = L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$

$$= \frac{1}{2}[L[1] - L[\cos 2t]]_{s\to s+3} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right].$$

PROBLEMS

Example Find $L[\cosh t \sin 2t]$. [Dec 2009]

Solution. $L[\cosh t \sin 2t] = L\left[\frac{e^t + e^{-t}}{2} \sin 2t\right] = \frac{1}{2}\left[L(e^t \sin 2t + e^{-t} \sin 2t)\right]$

$$= \frac{1}{2}\left[L[\sin 2t]_{s \to s-1} + L[\sin 2t]_{s \to s+1}\right]$$

$$= \frac{1}{2}\left[\left[\frac{2}{s^2+4}\right]_{s \to s-1} + \left[\frac{2}{s^2+4}\right]_{s \to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{(s-1)^2+4} + \frac{1}{(s+1)^2+4}.$$

Example Find $L[e^{2t} \cos 5t]$. [May 2008]

Solution. $L[e^{2t} \cos 5t] = L[\cos 5t]_{s \to s-2} = \left[\frac{s}{s^2+25}\right]_{s \to s-2} = \frac{s-2}{(s-2)^2+25}.$

PROBLEMS

Find the Laplace transforms of (i) $\cosh at \sinh bt$ (ii) $(1 + te^{-t})^3$.

Solution.

(i) $$L[\cosh at \sinh bt] = L\left[\frac{e^{at} + e^{-at}}{2} \sinh bt\right] = \frac{1}{2} L[e^{at} \sinh bt + e^{-at} \sinh bt]$$

$$= \frac{1}{2}\left[L[e^{at} \sinh bt] + L[e^{-at} \sinh bt]\right]$$

$$= \frac{1}{2}\left[L[\sinh bt]_{s \to s-a} + L[\sinh bt]_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\left[\frac{b}{s^2 + b^2}\right]_{s \to s-a} + \left[\frac{b}{s^2 + b^2}\right]_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{b}{(s-a)^2 + b^2} + \frac{b}{(s+a)^2 + b^2}\right].$$

(ii) $$L[(1 + te^{-t})^3] = L[1 + 3te^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}]$$

$$= L[1] + 3L[te^{-t}] + 3L[t^2 e^{-2t}] + L[t^3 e^{-3t}]$$

$$= \frac{1}{s} + 3L[t]_{s \to s+1} + 3L[t^2]_{s \to s+2} + L[t^3]_{s \to s+3}$$

$$= \frac{1}{s} + 3\left[\frac{1}{s^2}\right]_{s \to s+1} + 3\left[\frac{2}{s^3}\right]_{s \to s+2} + \left[\frac{6}{s^4}\right]_{s \to s+3}$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}.$$

PROBLEMS

Find $L[\cosh at \cos at]$. [Dec 2008]

Solution. $L[\cosh at \cos at] = \left[\frac{e^{at}+e^{-at}}{2}\cos at\right] = \frac{1}{2}\left[L[e^{at}\cos at] + L[e^{-at}\cos at]\right]$

$$= \frac{1}{2}\left[L[\cos at]_{s\to s-a} + L[\cos at]_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+a^2}\right)_{s\to s-a} + \left(\frac{s}{s^2+a^2}\right)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s-a}{(s-a)^2+a^2}\right) + \left(\frac{s+a}{(s+a)^2+a^2}\right)\right].$$

Example Find $L[e^{-3t}(2\cos 5t - 3\sin 5t)]$. [May 1999]

Solution. $L[e^{-3t}(2\cos 5t - 3\sin 5t)] = L[2e^{-3t}\cos 5t - 3e^{-3t}\sin 5t]$

$$= 2L[e^{-3t}\cos 5t] - 3L[e^{-3t}\sin 5t]$$

$$= 2L[\cos 5t]_{s\to s+3} - 3L[\sin 5t]_{s\to s+3}$$

$$= 2\left[\frac{s}{s^2+25}\right]_{s\to s+3} - 3\left[\frac{5}{s^2+25}\right]_{s\to s+3}$$

$$= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}.$$

PROPERTIES OF LALACE TRANSFORMS

Theorem. If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

Proof. We have $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Differentiating w.r.t s we get,

$$\frac{d}{ds}[F(s)] = \int_0^\infty \frac{d}{ds}[e^{-st} f(t)]dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)dt$$

$$-\frac{d}{ds}[F(s)] = \int_0^\infty e^{-st} t f(t)dt = L[t f(t)].$$

Now $L[t^2 f(t)] = L[t\, t f(t)] = -\frac{d}{ds} L[t f(t)]$

$$= -\frac{d}{ds}\Big(-\frac{d}{ds}(F(s))\Big) = (-1)^2 \frac{d^2}{ds^2} F(s)$$

In general, $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\big[F(s)\big]$.

PROBLEMS

Example Find $L[t \cosh 3t]$. [May 1995]

Solution. $L[t \cosh 3t] = -\frac{d}{ds}\{L[\cosh 3t]\} = -\frac{d}{ds}\left[\frac{s}{s^2-9}\right] = -\left[\frac{s^2+9-s\times 2s}{(s^2-9)^2}\right]$

$$= -\frac{9-s^2}{(s^2-9)^2} = \frac{s^2-9}{(s^2-9)^2} = \frac{1}{(s^2-9)}.$$

Example Find $L[t^2 \sin at]$. [May 2001]

Solution. $L[t^2 \sin at] = (-1)^2 \frac{d^2}{ds^2}[L[\sin at]] = \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$

$$= a\frac{d^2}{ds^2}(s^2+a^2)^{-1} = a\frac{d}{ds}[(-1)(s^2+a^2)^{-2}2s] = -2a\frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= -2a\left(\frac{(s^2+a^2)^2.1 - s.2.(s^2+a^2)2s}{(s^2+a^2)^4}\right)$$

$$= -2a\frac{(s^2+a^2)\left[s^2+a^2-4s^2\right]}{(s^2+a^2)^4} = -2a\frac{[-3s^2+a^2]}{(s^2+a^2)^3} = 2a\frac{[3s^2-a^2]}{(s^2+a^2)^3}.$$

PROBLEMS

Example Find $L[t \cos 3t]$.

Solution. $L[t \cos 3t] = -\frac{d}{ds}\{L[\cos 3t]\} = -\frac{d}{ds}\left[\frac{s}{s^2+9}\right]$

$$= -\left[\frac{s^2+9-s2s}{(s^2+9)^2}\right] = -\frac{9-s^2}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}.$$

Example Find Laplace transform of $t \sin 2t$. [Dec 2010]

Solution. $L[t \sin 2t] = -\frac{d}{ds}\{L[\sin 2t]\}$

$$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\frac{d}{ds}\left[2 \cdot (s^2+4)^{-1}\right]$$

$$= -2 \cdot (-1) \cdot (s^2+4)^{-2} \cdot 2s$$

$$= \frac{4s}{(s^2+4)^2}.$$

Example Find $L[t \sin at]$.

Solution. $L[t \sin at] = -\frac{d}{ds}\{L[\sin at]\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -a\frac{d}{ds}(s^2+a^2)^{-1}$

$$= -a(-1)(s^2+a^2)^{-2}.2s = \frac{2as}{(s^2+a^2)^2}.$$

PROBLEMS

Find $L[t\cos at]$.

Solution. $L[t\cos at] = -\frac{d}{ds}\big[L[\cos at]\big] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] = -\frac{s^2+a^2-s2s}{(s^2+a^2)^2}$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Example Find $L[\sin at - at\cos at]$. [May 2003]

Solution. $L[\sin at - at\cos at] = L[\sin at] - aL[t\cos at]$

$$= \frac{a}{s^2+a^2} - a\left(-\frac{d}{ds}L[\cos at]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{(s^2+a^2).1 - s2s}{(s^2+a^2)^2}$$

$$= \frac{a}{s^2+a^2} + a\frac{s^2+a^2-2s^2}{(s^2+a^2)^2} = \frac{a}{s^2+a^2} + \frac{a(a^2-s^2)}{(s^2+a^2)^2} = a\left[\frac{s^2+a^2+a^2-s^2}{(s^2+a^2)^2}\right] = \frac{2a^3}{(s^2+a^2)^2}.$$

PROBLEMS

Example Find $L[te^{-t}\sin t]$.

Solution. $L[te^{-t}\sin t] = -\frac{d}{ds}\{L[e^{-t}\sin t\} = -\frac{d}{ds}\left[L[\sin t]_{s\to s+1}\right]$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]_{s\to s+1} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right] = -(-1)(s^2+2s+2)^{-2}(2s+2)$$

$$= \frac{2s+2}{(s^2+2s+2)^2}.$$

Example Find $L[t\sin 3t\cos 2t]$. [May 2008]

Solution. $L[t\sin 3t\cos 2t] = L\left[t\frac{[\sin 5t+\sin t]}{2}\right]$

$$= \frac{1}{2}\left(-\frac{d}{ds}\right)\left(L[\sin 5t] + L[\sin t]\right)$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[5(s^2+25)^{-1} + (s^2+1)^{-1}1\right]$$

$$= -\frac{1}{2}\left[\frac{-5.\,2s}{(s^2+25)^2} - \frac{2s}{(s^2+1)^2}\right]$$

$$= -\frac{1}{2}(-2)\left[\frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}.$$

PROBLEMS

Example Find $L[t^2e^{-3t}\sin 2t]$. [Jun 2013, May 2000]

Solution. $L[t^2e^{-3t}\sin 2t] = \frac{d^2}{ds^2}\left[L[e^{-3t}\sin 2t]\right] = \frac{d^2}{ds^2}\left[L[\sin 2t]_{s\to s+3}\right]$

$$= \frac{d^2}{ds^2}\left[\frac{2}{(s+3)^2+4}\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{s^2+6s+13}\right]$$

$$= 2\frac{d^2}{ds^2}(s^2+6s+13)^{-1}$$

$$= 2\frac{d}{ds}\left[(-1)(s^2+6s+13)^{-2}(2s+6)\right]$$

$$= -2\frac{d}{ds}\left[\frac{2s+6}{(s^2+6s+13)^2}\right]$$

$$= -4\frac{d}{ds}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$$

$$= -4\frac{(s^2+6s+13)^2-(s+3)2(s^2+6s+13)(2s+6)}{(s^2+6s+13)^4}$$

$$= -4\frac{s^2+6s+13-4(s^2+6s+9)}{(s^2+6s+13)^3}$$

$$= -4\frac{s^2+6s+13-4s^2-36-24s}{(s^2+6s+13)^3}$$

$$= -4\frac{-3s^2-18s-23}{(s^2+6s+13)^3} = 4\frac{3s^2+18s+23}{(s^2+6s+13)^3}.$$

PROPERTIES OF LALACE TRANSFORMS

Theorem. If $L[f(t)] = F[s]$ and if $\lim_{t \to 0} \frac{f(t)}{t}$ exists, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$.

Proof. We have, $F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

Integrating both sides w.r.t s from s to ∞

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty e^{-st} f(t)dtds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt = -\int_0^\infty \frac{f(t)}{t}(e^{-\infty} - e^{-st})dt = \int_0^\infty e^{-st}\frac{f(t)}{t}dt = L\left[\frac{f(t)}{t}\right].$$

PROBLEMS

Example Find $L\left[\frac{1-e^t}{t}\right]$. [Dec 2008]

Solution. $L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L[1-e^t]ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$

$$= \left[\log s - \log(s-1)\right]_s^\infty = \log\left(\frac{s}{s-1}\right)_s^\infty$$

$$= \log\left(\frac{1}{1-\frac{1}{s}}\right)_s^\infty = \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= \log\left(\frac{s-1}{s}\right).$$

Example Find the Laplace transform of $\frac{1-e^{-t}}{t}$. [Dec 2013]

Solution. $L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L[1-e^{-t}]ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds$

$$= \left[\log s - \log(s+1)\right]_s^\infty = \left[\log \frac{s}{s+1}\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1+\frac{1}{s}}\right)\right]_s^\infty = \log 1 - \log\left(\frac{s}{s+1}\right)$$

$$= \log\left(\frac{s+1}{s}\right).$$

PROBLEMS

Example Find $L\left[\frac{\sin 2t}{t}\right]$. [Jan 2006]

Solution. $L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty L[\sin 2t]ds = \int_s^\infty \frac{2}{s^2+4}ds$

$$= 2\frac{1}{2}\left(\tan^{-1}\frac{s}{2}\right)_s^\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right).$$

PROBLEMS

Example Find $L\left[\frac{1-\cos 2t}{t}\right]$. [Apr 2004]

Solution. $L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty L[1-\cos 2t]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$

$$= \log s - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \log \frac{\sqrt{s^2+4}}{s}.$$

Example Find $L\left[\frac{\sin at}{t}\right]$. [Dec 2009]

Solution. $L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]ds = \int_s^\infty \frac{a}{s^2+a^2}ds$

$$= a\frac{1}{a}\left(\tan^{-1}\frac{s}{a}\right)_s^\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right).$$

PROBLEMS

Find $L\left[\frac{\cos at - \cos bt}{t}\right]$. [Dec 2012, May 2011, Dec 2007]

Solution. $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L\left[\cos at - \cos bt\right]ds$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}.$$

Example Find $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$. [May 2006]

Solution. $L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty L\left[e^{-at} - e^{-bt}\right]ds = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \log\frac{s+b}{s+a}.$$

PROBLEMS

Find the Laplace transform of $\frac{e^{at}-e^{-bt}}{t}$. [Jun 2012]

Solution. $L\left[\frac{e^{at}-e^{-bt}}{t}\right] = \int_s^{\infty} L\left[e^{at}-e^{-bt}\right] ds = \int_s^{\infty}\left(\frac{1}{s-a}-\frac{1}{s+b}\right) ds$

$$= \left[\log(s-a)-\log(s+b)\right]_s^{\infty} = \left[\log\frac{s-a}{s+b}\right]_s^{\infty}$$

$$= \log\frac{s+b}{s-a}.$$

Example Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$. [May 2007]

Solution. $L\left[\frac{e^{-3t}\sin 2t}{t}\right] = \int_s^{\infty} L\left[e^{-3t}\sin 2t\right] ds = \int_s^{\infty} L\left[\sin 2t\right]_{s\to s+3} ds$

$$= \int_s^{\infty} \frac{2}{(s+3)^2+2^2} ds$$

$$= 2\frac{1}{2}\left(\tan^{-1}\frac{s+3}{2}\right)_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)$$

$$= \cot^{-1}\left(\frac{s+3}{2}\right).$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

Laplace transform of periodic functions

A function $f(t)$ is said to be periodic if there exists a positive constant T such that $f(t+T) = f(t)$ for all t.The smallest of such T is called the period of the function.

Example. $\sin(t+2\pi) = \sin t$. Sine function is a periodic function with period 2π.

Theorem. If $f(t)$ is a periodic function with period T, then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

Example Find the Laplace transform of the rectangular wave function given by $f(t) = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \end{cases}$ with $f(t + 2b) = f(t)$. [May 2009]

Solution. Since $f(t + 2b) = f(t)$, it is a periodic function with period $2b$.

$$\therefore L[f(t)] = \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt = \frac{1}{1 - e^{-2bs}} \left[\int_0^{b} e^{-st} f(t) dt + \int_b^{2b} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1 - e^{-2bs}} \left[\int_0^{b} e^{-st} dt - \int_b^{2b} e^{-st} dt \right]$$

$$= \frac{1}{1 - e^{-2bs}} \left[\left(\frac{e^{-st}}{-s} \right)_0^b - \left(\frac{e^{-st}}{-s} \right)_b^{2b} \right]$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

$$= \frac{1}{1-e^{-2bs}}\left(-\frac{1}{s}(e^{-bs}-1)+\frac{1}{s}(e^{-2bs}-e^{-bs})\right)$$

$$= \frac{1}{s(1-e^{-2bs})}\left(-e^{-bs}+1+e^{-2bs}-e^{-bs}\right)$$

$$= \frac{1}{s(1-e^{-2bs})}\left(1-2e^{-bs}+e^{-2bs}\right)$$

$$= \frac{1}{s(1-(e^{-bs})^2)}\left(1-2e^{-bs}+(e^{-bs})^2\right)$$

$$= \frac{(1-e^{-bs})^2}{s(1-e^{-bs})(1+e^{-bs})} = \frac{1-e^{-bs}}{s(1+e^{-bs})} = \frac{e^{\frac{bs}{2}}-e^{-\frac{bs}{2}}}{s\left(e^{\frac{bs}{2}}+e^{-\frac{bs}{2}}\right)} = \frac{1}{s}\tan h\left(\frac{bs}{2}\right).$$

Example Find the Laplace transform of the square-wave function (or Meoander function) of period a defined as $f(t) = \begin{cases} 1 & \text{when } 0 < t < \frac{a}{2} \\ -1 & \text{when } \frac{a}{2} < t < a. \end{cases}$ [Jun 2013]

Solution. Since $f(t)$ is a periodic function of period a, we have

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} f(t)dt + \int_{\frac{a}{2}}^a e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} dt + \int_{\frac{a}{2}}^a e^{-st}(-1)dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s}\right)_{\frac{a}{2}}^a \right]$$

$$= \frac{1}{s(1-e^{-as})} \left[-\left(e^{-\frac{a}{2}s} - 1\right) + e^{-as} - e^{-\frac{a}{2}s} \right]$$

$$= \frac{1}{s(1-e^{-as})} \left[1 - 2e^{-\frac{a}{2}s} + e^{-as} \right]$$

$$= \frac{\left(1 - e^{-\frac{a}{2}s}\right)^2}{s\left(1 - e^{-\frac{a}{2}s}\right)\left(1 + e^{-\frac{a}{2}s}\right)}$$

$$= \frac{1 - e^{-\frac{a}{2}s}}{s\left(1 + e^{-\frac{a}{2}s}\right)}$$

$$= \frac{1}{s} \cdot \frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}}$$

$$= \frac{1}{s} \tan h\left(\frac{as}{4}\right).$$

Example nd the Laplace transform of a square wave function given by

$$f(t) = \begin{cases} \epsilon & \text{for } 0 \le t \le \frac{a}{2} \\ -\epsilon & \text{for } \frac{a}{2} \le t \le a \end{cases}$$

and $f(t+a) = f(t)$. [Dec 2011]

Solution. Since $f(t+a) = f(t)$, $f(t)$ is a periodic function of period a.

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} f(t)dt + \int_{\frac{a}{2}}^a e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} \epsilon dt + \int_{\frac{a}{2}}^a e^{-st} (-\epsilon) dt \right]$$

$$= \frac{\epsilon}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s} \right)_{\frac{a}{2}}^a \right]$$

$$= \frac{\epsilon}{s(1-e^{-as})} \left[-\left(e^{-\frac{sa}{2}} - 1\right) + e^{-as} - e^{-\frac{as}{2}} \right]$$

$$= \frac{\epsilon}{s(1-e^{-as})} \left[1 - 2e^{-\frac{as}{2}} + e^{-as} \right]$$

$$= \frac{\epsilon \left(1 - e^{-\frac{as}{2}}\right)^2}{s\left(1 + e^{-\frac{as}{2}}\right)\left(1 - e^{-\frac{as}{2}}\right)}$$

$$= \frac{\epsilon}{s} \frac{\left(1 - e^{-\frac{as}{2}}\right)}{\left(1 + e^{-\frac{as}{2}}\right)}$$

$$= \frac{\epsilon}{s} \cdot \frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}}$$

$$= \frac{\epsilon}{s} \tan h \left(\frac{as}{4} \right).$$

Example Find the Laplace transform of $f(t) = \begin{cases} \epsilon & 0 \le t \le a \\ -\epsilon & a \le t \le 2a \end{cases}$

and $f(t+2a) = f(t)$ for all t. [Dec 2010]

Solution. Since $f(t+2a) = f(t)$ for all t, $f(t)$ is a periodic function with period $2a$.

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} \epsilon dt + \int_a^{2a} e^{-st} (-\epsilon)dt \right]$$

$$= \frac{\epsilon}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{\epsilon}{s(1-e^{-2as})} \left[-(e^{-as} - 1) + e^{-2as} - e^{-as} \right]$$

$$= \frac{\epsilon \left(1 - 2e^{-as} + e^{-2as} \right)}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{\epsilon (1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{\epsilon}{s} \frac{(1-e^{-as})}{(1+e^{-as})} = \frac{\epsilon}{s} \frac{\left(e^{\frac{as}{2}} - e^{-\frac{as}{2}} \right)}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}} \right)}$$

$$= \frac{\epsilon}{s} \tan h \left(\frac{as}{2} \right).$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

Example Find the Laplace transform of the triangular wave function defined by $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a - t & a < t \le 2a \end{cases}$ and $f(t)$ is of period $2a$. [May 2015,May 2011]

Solution. Since $f(t)$ is of period $2a$,we have

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} t dt + \int_a^{2a} e^{-st}(2a-t)dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t d\left(\frac{e^{-st}}{-s}\right) + \int_a^{2a} (2a-t) d\left(\frac{e^{-st}}{-s}\right)\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left(t\frac{e^{-st}}{-s}\right)_0^a - \int_0^a \frac{e^{-st}}{-s}dt + \left((2a-t)\frac{e^{-st}}{-s}\right)_a^{2a} - \int_a^{2a} \frac{e^{-st}}{-s}(-dt)\right]$$

$$= \frac{1}{1-e^{-2as}}\left[a\frac{e^{-as}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^a + 0 + \frac{a}{s}e^{-as} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}(e^{-as}-1) + \frac{a}{s}e^{-as} + \frac{1}{s^2}(e^{-2as}-e^{-as})\right]$$

$$= \frac{1}{1-e^{-2as}}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + a\frac{e^{-as}}{s} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{e^{-2as}-e^{-as}-e^{-as}+1}{s^2}\right] = \frac{1}{1-e^{-as}}\left[\frac{1-2e^{-2as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\cdot\frac{(1-e^{-as})^2}{s^2} = \frac{(1-e^{-as})}{s^2.(1+e^{-as})}$$

$$= \frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{s^2\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)} = \frac{1}{s^2}\tan h\left(\frac{as}{2}\right).$$

Example Find the Laplace transform of the half-sine wave rectifier function defined by $f(t) = \begin{cases} \sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases}$ [Jun 2014, Dec 2012]

Solution. $f(t)$ is defined in the interval $(0, \frac{2\pi}{\omega})$

$$f(t + \frac{2\pi}{\omega}) = \sin\omega(t + \frac{2\pi}{\omega}) = \sin(\omega t + 2\pi) = \sin\omega t = f(t)$$

$\therefore f(t)$ is periodic with period $T = \dfrac{2\pi}{\omega}$.

$$\text{We know that}, L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt = \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\sin\omega t dt$$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\left[\frac{e^{-st}}{s^2+w^2}[-s\sin\omega t - \omega\cos\omega t]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{s2\pi}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}}{s^2+w^2}[-s.0 - \omega(-1)] - \frac{-\omega}{\omega^2+s^2}\right]$$

$$= \frac{1}{(1-e^{-\frac{s2\pi}{\omega}})(s^2+\omega^2)}\left\{\omega e^{\frac{-s\pi}{\omega}} + \omega\right\}$$

$$= \frac{\omega(1+e^{-\frac{s\pi}{\omega}})}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})(1+e^{\frac{-s\pi}{\omega}})} = \frac{\omega}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})}.$$

Example Find the Laplace transform of the saw-toothed wave function of period T given by $f(t) = \frac{t}{T}, 0 < t < T$.

Solution. Since $f(t)$ is a periodic function of period T, we have

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{t}{T} dt = \frac{1}{T(1-e^{-sT})} \int_0^T t d\left(\frac{e^{-st}}{-s}\right)$$

$$= \frac{1}{T(1-e^{-sT})}\left[\left(\frac{te^{-st}}{-s}\right)_0^T - \int_0^T \frac{e^{-st}}{-s} dt\right]$$

$$= \frac{1}{T(1-e^{-sT})}\left[\frac{Te^{-sT}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^T\right]$$

$$= \frac{1}{sT(1-e^{-sT})}\left[-Te^{-sT} - \frac{1}{s}(e^{-sT}-1)\right]$$

$$= \frac{1}{sT(1-e^{-sT})}\left[\frac{1}{s} - \frac{1}{s}e^{-sT} - Te^{-sT}\right]$$

$$= \frac{1}{sT(1-e^{-sT})}\left[\frac{1-e^{-sT}}{s} - Te^{-sT}\right]$$

$$= \frac{1}{Ts^2} - \frac{e^{-sT}}{s(1-e^{-sT})}.$$

INITIAL AND FINAL VALUE THEOREM

Initial value theorem. If the Laplace transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Proof. By Laplace transform of derivative of $f(t)$ we have

$$L[f'(t)] = sL[f(t)] - f(0).$$

$$\int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0).$$

Taking limit as $s \to \infty$ we obtain.

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)dt = \lim_{s\to\infty} sF(s) - f(0).$$

$$\int_0^\infty \lim_{s\to\infty} e^{-st} f(t)dt = \lim_{s\to\infty} sF(s) - f(0).$$

$$0 = \lim_{s\to\infty} sF(s) - f(0).$$

$$f(0) = \lim_{s\to\infty} sF(s).$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Final value theorem. If the Laplace transform of $f(t)$ and $f'(t)$ exist and $F(s) = L[f(t)]$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$. [Dec 2014]

Proof. We have $L[f'(t)] = sL[f(t)] - f(0)$.

ie., $\int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0)$.

Taking limit as $s \to 0$ we obtain.

$$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$\int_0^\infty \lim_{s\to 0} e^{-st} f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$\int_0^\infty f'(t)dt = \lim_{s\to 0} sF(s) - f(0).$$

$$[f(t)]_0^\infty = \lim_{s\to 0} sF(s) - f(0).$$

$$f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

UNIT STEP FUNCTION

Unit Step Function. The unit step function u is defined as

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

$u(t)$ has jump discontinuity at $t = 0$.

More generally we define $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$

Laplace transform of unit step function [Jun 2010]

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)dt = \int_0^a e^{-st} u(t-a)dt + \int_a^\infty e^{-st} u(t-a)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}dt = 0 + \left(\frac{e^{-st}}{-s}\right)_a^\infty$$

$$= \frac{-1}{s}[0 - e^{-as}] = \frac{e^{-as}}{s} \text{ if } s > 0.$$

Second Shifting property

If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{-as}F(s) = e^{-as}L[f(t)]$.

Proof. We have $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$

UNIT STEP FUNCTION

Now $f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a. \end{cases}$

$$\therefore L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)dt + \int_a^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} 0 dt + \int_a^\infty e^{-st} f(t-a)dt$$

Let $t - a = x$ When $t = a, x = 0.$

$dt = dx$ When $t = \infty, x = \infty.$

$$\therefore L[f(t-a)u(t-a)] = \int_0^\infty e^{-s(a+x)} f(x)dx = \int_0^\infty e^{-as-sx} f(x)dx = \int_0^\infty e^{-as} e^{-sx} f(x)dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)dx = e^{-as} \int_0^\infty e^{-st} f(t)dt = e^{-as} F(s).$$

UNIT IMPULSE FUNCTION

Note. The second shifting property can be stated as follows.

If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{-as}L[f(t)]$ where $u(t-a)$ is the unit step function.

The unit impulse function. For any positive ϵ, the impulse function δ_ϵ is defined as $\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}(u(t) - u(t-\epsilon)), & 0 \le t < \epsilon \\ 0, & \text{Otherwise} \end{cases}$

Dirac delta function. $\lim_{\epsilon \to 0} \delta_\epsilon(t)$ is called the Dirac delta function, denoted by $\delta(t)$.

$$\therefore \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

In general $\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a. \end{cases}$

RESULTS

Results. 1. Find $L[\delta_\epsilon(t-a)]$.

Solution

By the definition, $\delta_\epsilon(t-a) = \frac{1}{\epsilon}(u(t-a) - u(t-a-\epsilon))$ if $a \le t \le a+\epsilon$ $[0 \le t-a < \epsilon]$.

$$\text{Now, } L[\delta_\epsilon(t-a)] = \frac{1}{\epsilon}\Big[L[u(t-a) - u(t-a-\epsilon)]\Big] = \frac{1}{\epsilon}\Big[L[u(t-a)] - L[u(t-a)-\epsilon]\Big]$$

$$= \frac{1}{\epsilon}\Big[L[u(t-a)] - L[u(t-(a+\epsilon)]\Big] = \frac{1}{\epsilon}\Big[\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\Big]$$

$$= \frac{1}{\epsilon}\Big[\frac{e^{-as}}{s} - \frac{e^{-as}.e^{-\epsilon s}}{s}\Big]$$

$$L[\delta_\epsilon(t-a)] = \frac{1}{\epsilon s}e^{-as}(1 - e^{-\epsilon s})$$

Taking limit as $\epsilon \to 0$ we obtain

$$L[\lim_{\epsilon \to 0} \delta_\epsilon(t-a)] = e^{-as} \lim_{\epsilon s \to 0} \frac{1 - e^{-\epsilon s}}{\epsilon s}$$

$$L[\delta(t-a)] = e^{-as}.1 \quad [\lim_{x \to 0} \frac{1-e^x}{x} = 1]$$

$$= e^{-as}.$$

2. When $a = 0, L[\delta(t)] = 1$.

3. One important property of Dirac Delta funtion is $\int_0^\infty f(t)\delta(t-a)dt = f(a)$.

RROBLEMS

Example If $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$ find $L[f(t)]$. [Apr 2010]

Solution. Let $g\left(t - \frac{2\pi}{3}\right) = f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$.

$$\text{Now } L[f(t)] = L\left[g\left(t - \frac{2\pi}{3}\right)\right] = e^{\frac{-2\pi s}{3}} L[g(t)] = e^{\frac{-2\pi s}{3}} L[\cos t] = e^{\frac{-2\pi}{3}s} \frac{s}{s^2+1}.$$

Example Find $L[(t-1)^2 u(t-1)]$.

Solution. $L[(t-1)^2 u(t-1)] = e^{-s} L[t^2] = e^{-s} \frac{2}{s^3} = \frac{2e^{-s}}{s^3}$.

Example Find $L[e^{-4t} u(t-1)]$.

Solution. $L[e^{-4t} u(t-1)] = L[e^{-4(t-1+1)} u(t-1)] = L[e^{-4(t-1)-4)} u(t-1)]$

$$= e^{-4} e^{-s} L[e^{-4t}] = e^{-4} e^{-s} \frac{1}{s+4} = \frac{e^{-(4+s)}}{s+4}.$$

PROBLEMS

Example Verify the initial value theorem for the function $2 + 3\cos t$.

Solution. $f(t) = 2 + 3\cos t$

$$L[f(t)] = L[2 + 3\cos t] = 2L[1] + 3L[\cos t]$$

$$= 2\frac{1}{s} + 3\frac{s}{s^2+1}$$

$$= \frac{2}{s} + \frac{3s}{s^2+1} = F(s)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(2 + 3\cos t) = 2 + 3\cos 0 = 2 + 3 = 5.$$

$$\therefore \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{2}{s} + \frac{3s}{s^2+1}\right) = \lim_{s\to\infty}\left(2 + \frac{3s^2}{s^2+1}\right)$$

$$= 2 + 3\lim_{s\to\infty}\left(\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right)$$

$$= 2 + 3\lim_{s\to\infty}\left(\frac{1}{1+\frac{1}{s^2}}\right) = 2 + 3 = 5.$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

PROBLEMS

Verify the initial and final value theorems for the function $f(t) = ae^{-bt}$. [Jun 2013, May 1997]

Solution. Let $f(t) = ae^{-bt}$.

$$L[f(t)] = \frac{a}{s+b} = F(s)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{as}{s+b} = \lim_{s\to\infty} \frac{as}{s(1+\frac{b}{s})} = a$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\lim_{s\to\infty} f(t) = \lim_{t\to 0} ae^{-bt} = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} ae^{-st} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{as}{s^2+a^2} = 0$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

PROBLEMS

Verify the initial value theorem for the function

$f(t) = 1 + e^{-t}(\sin t + \cos t)$. [Jun 2012, Dec 2010, Jun 2010]

Solution. Let $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

$$L[f(t)] = L[1 + e^{-t}(\sin t + \cos t)]$$

$$= L[1] + L[e^{-t}(\sin t + \cos t)]$$

$$= \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$$

$$= \frac{1}{s} + L\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{1}{s} + \left[\frac{s+1}{s^2+1}\right]_{s\to s+1}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[1 + e^{-t}(\sin t + \cos t)] = 1 + 1 = 2.$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1 + \lim_{s\to\infty}\frac{s^2(1+\frac{2}{s})}{s^2(1+\frac{1}{s})^2 + \frac{1}{s^2}} = 1 + \frac{1}{1} = 2.$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

STANDARD RESULTS

Standard Results

1. $L[1] = \frac{1}{s} \Rightarrow L^{-1}\left[\frac{1}{s}\right] = 1.$

2. $L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$

3. $L[e^{-at}] = \frac{1}{s+a} \Rightarrow L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}.$

4. $L[t] = \frac{1}{s^2} \Rightarrow L^{-1}\left[\frac{1}{s^2}\right] = t.$

5. $L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n.$

6. $L[\cosh at] = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at.$

STANDARD RESULTS

7. $L[\sinh at] = \dfrac{a}{s^2 - a^2} \Rightarrow L^{-1}\left[\dfrac{a}{s^2 - a^2}\right] = \sinh at.$

8. $L[\cos at] = \dfrac{s}{s^2 + a^2} \Rightarrow L^{-1}\left[\dfrac{s}{s^2 + a^2}\right] = \cos at.$

9. $L[\sin at] = \dfrac{a}{s^2 + a^2} \Rightarrow L^{-1}\left[\dfrac{a}{s^2 + a^2}\right] = \sin at.$

10. $L[t \sin at] = \dfrac{2as}{(s^2 + a^2)^2} \Rightarrow L^{-1}\left[\dfrac{2as}{(s^2 + a^2)^2}\right] = t \sin at.$

11. $L[t \cos at] = \dfrac{s^2 - a^2}{(s^2 + a^2)^2} \Rightarrow L^{-1}\left[\dfrac{s^2 - a^2}{(s^2 + a^2)^2}\right] = t \cos at.$

PROPERTIES OF INVERSE LT

Basic theorems

1. $L^{-1}[aF(s)+bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$ where a and b are constants.

Shifting theorems

2. (i) If $L[f(t)] = F(s)$, then

$$L[e^{-at}f(t)] = F(s+a)$$

$$\implies L^{-1}[F(s+a)] = e^{-at}f(t) = e^{-at}L^{-1}[F(s)]$$

(ii) $L[e^{at}f(t)] = F(s-a)$

$$\implies L^{-1}[F(s-a)] = e^{at}f(t) = e^{at}L^{-1}[F(s)].$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$. [Jan 2008]

Solution. $L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$

$$= L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{s+4}\right]+L^{-1}\left[\frac{1}{s^2+4}\right]+L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= t+e^{-4t}+\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]+\cosh 3t$$

$$= t+e^{-4t}+\frac{1}{2}\sin 2t+\cosh 3t.$$

Example Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$.

Solution. $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]=L^{-1}\left[\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right]=L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s^2}\right]+4L^{-1}\left[\frac{1}{s^3}\right]$

$$= 1-3t+\frac{4}{2}L^{-1}\left[\frac{2}{s^3}\right]=1-3t+2t^2.$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{3s+5}{s^2+9}\right]$.

Solution. $L^{-1}\left[\frac{3s+5}{s^2+9}\right] = 3L^{-1}\left[\frac{s}{s^2+9}\right] + 5L^{-1}\left[\frac{1}{s^2+9}\right] = 3\cos 3t + \frac{5}{3}\sin 3t$

$$= \frac{9\cos 3t + 5\sin 3t}{3}.$$

Example Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$.

Solution. $L^{-1}\left[\frac{3s+2}{s^2-4}\right] = 3L^{-1}\left[\frac{s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right] = 3\cosh 2t + \sinh 2t.$

Example Find $L^{-1}\left[\frac{s}{a^2s^2+b^2}\right]$.

Solution. $L^{-1}\left[\frac{s}{a^2s^2+b^2}\right] = L^{-1}\left[\frac{s}{a^2(s^2+\frac{b^2}{a^2})}\right] = \frac{1}{a^2}L^{-1}\left[\frac{s}{s^2+\frac{b^2}{a^2}}\right] = \frac{1}{a^2}\cos\left(\frac{b}{a}t\right).$

Example Find $L^{-1}\left[\frac{1}{(s-3)^5}\right]$.

Solution. $L^{-1}\left[\frac{1}{(s-3)^5}\right] = e^{3t}L^{-1}\left[\frac{1}{s^5}\right] = \frac{e^{3t}}{4!}t^4 = \frac{e^{3t}t^4}{24}.$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{s}{(s+6)^3}\right]$.

Solution. $L^{-1}\left[\frac{s}{(s+6)^3}\right] = L^{-1}\left[\frac{s+6-6}{(s+6)^3}\right] = e^{-6t}L^{-1}\left[\frac{s-6}{s^3}\right] = e^{-6t}\left[L^{-1}\left[\frac{1}{s^2}\right] - 6L^{-1}\left[\frac{1}{s^3}\right]\right]$

$$= e^{-6t}\left[t - \frac{6}{2}t^2\right] = e^{-6t}[t - 3t^2].$$

Example Find $L^{-1}\left[\frac{s+3}{s^2-4s+13}\right]$.

Solution. $L^{-1}\left[\frac{s+3}{s^2-4s+13}\right] = L^{-1}\left[\frac{s+3}{(s-2)^2+13-4}\right] = L^{-1}\left[\frac{s+3}{(s-2)^2+9}\right]$

$$= L^{-1}\left[\frac{s-2+2+3}{(s-2)^2+9}\right]$$

$$= e^{2t}\left\{L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{5}{s^2+9}\right]\right\}$$

$$= e^{2t}\left[\cos 3t + \frac{5}{3}\sin 3t\right].$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{s}{s^2-4s+5}\right]$.

Solution. $L^{-1}\left[\frac{s}{s^2-4s+5}\right] = L^{-1}\left[\frac{s}{(s-2)^2+5-4}\right] = L^{-1}\left[\frac{s-2+2}{(s-2)^2+1}\right]$

$$= e^{2t}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$= e^{2t}\left[L^{-1}\left[\frac{s}{s^2+1}\right] + 2L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= e^{2t}\left[\cos t + 2\sin t\right].$$

Example Find $L^{-1}\left[\frac{1}{s^2+4s+2}\right]$. [Dec 2010]

Solution. $L^{-1}\left[\frac{1}{s^2+4s+2}\right] = L^{-1}\left[\frac{1}{(s+2)^2+2-4}\right] = L^{-1}\left[\frac{1}{(s+2)^2-2}\right]$

$$= e^{-2t}L^{-1}\left[\frac{1}{s^2-(\sqrt{2})^2}\right]$$

$$= \frac{e^{-2t}}{\sqrt{2}}L^{-1}\left[\frac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right]$$

$$= \frac{e^{-2t}}{\sqrt{2}}\sin(\sqrt{2}t).$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right]$.

Solution. $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{2s-3}{(s+2)^2+13-4}\right] = L^{-1}\left[\frac{2(s+2)-7}{(s+2)^2+9}\right]$

$$= e^{-2t}L^{-1}\left[\frac{2s-7}{s^2+9}\right] = e^{-2t}\left[2L^{-1}\left[\frac{s}{s^2+9}\right] - 7L^{-1}\left[\frac{1}{s^2+9}\right]\right]$$

$$= e^{-2t}\left[2\cos 3t - \frac{7}{3}\sin 3t\right] = \frac{e^{-2t}}{3}\left[6\cos 3t - 7\sin 3t\right].$$

Example Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$. [May 2007]

Solution. $L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right] = e^{-2t}L^{-1}\left[\frac{s-2}{s^2+1}\right]$

$$= e^{-2t}\left[L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= e^{-2t}[\cos t - 2\sin t].$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{s}{(s+3)^2}\right]$. [Dec 2009]

Solution. $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right] = e^{-3t}L^{-1}\left[\frac{s-3}{s^2}\right]$

$$= e^{-3t}\left[L^{-1}\left[\frac{1}{s} - \frac{3}{s^2}\right]\right] = e^{-3t}\left[L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{3}{s^2}\right]\right]$$

$$= e^{-3t}[1-3t].$$

Example Find $L^{-1}\left[\frac{1}{(s-4)^5} + \frac{5}{(s-2)^2+25} + \frac{s+3}{(s+3)^2+36}\right]$. [Dec 2007]

Solution. $L^{-1}\left[\frac{1}{(s-4)^5} + \frac{5}{(s-2)^2+25} + \frac{s+3}{(s+3)^2+36}\right]$

$$= L^{-1}\left[\frac{1}{(s-4)^5}\right] + L^{-1}\left[\frac{5}{(s-2)^2+25}\right] + L^{-1}\left[\frac{s+3}{(s+3)^2+36}\right]$$

$$= e^{4t}L^{-1}\left[\frac{1}{s^5}\right] + e^{2t}L^{-1}\left[\frac{5}{s^2+25}\right] + e^{-3t}L^{-1}\left[\frac{s}{s^2+36}\right]$$

$$= e^{4t}\frac{t}{4!} + e^{2t}\sin 5t + e^{-3t}\cos 6t$$

$$= e^{4t}\frac{t}{24} + e^{2t}\sin 5t + e^{-3t}\cos 6t.$$

INVERSE LT BY PARTIAL FRACTION METHOD

Example Find $L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]$. [May 2008]

Solution. Let $\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$

$$1 = A(s+3) + B(s+1)$$

Put $s = -1, 1 = 2A \Rightarrow A = \frac{1}{2}$.

Put $s = -3, 1 = -2B \Rightarrow B = \frac{-1}{2}$.

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{2}(e^{-t} - e^{-3t}).$$

INVERSE LT BY PARTIAL FRACTION METHOD

Example Find $L^{-1}\left[\frac{1}{s(s+3)^3}\right]$. [Nov 2005]

Solution. Let $\frac{1}{s(s+3)^3} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{(s+3)^2} + \frac{D}{(s+3)^3}$.

$$1 = A(s+3)^3 + Bs(s+3)^2 + Cs(s+3) + Ds.$$

When $s = 0, 1 = 27A \Rightarrow A = \frac{1}{27}$.

When $s = -3, 1 = -3D \Rightarrow B = \frac{-1}{3}$.

Equating the coeff. of s^3 we get

$A + B = 0 \Rightarrow B = -A = \frac{-1}{27}$.

Equating the coeff. of s^2 w

$9A + 6B + C = 0$

$9 \times \frac{1}{27} + 6\left(\frac{-1}{27}\right) + C = 0$

$\frac{9}{27} - \left(\frac{6}{27}\right) + C = 0$

$\frac{3}{27} + C = 0$

$C = -\frac{1}{9}$.

$$\therefore \frac{1}{s(s+3)^3} = \frac{\frac{1}{27}}{s} - \frac{\frac{1}{27}}{s+3} - \frac{\frac{1}{9}}{(s+3)^2} + \frac{\frac{1}{3}}{(s+3)^3}$$

$$L^{-1}\left[\frac{1}{s(s+3)^3}\right] = \frac{1}{27}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{27}\left[\frac{1}{s+3}\right] - \frac{1}{9}\left[\frac{1}{(s+3)^2}\right] - \frac{1}{3}\left[\frac{1}{(s+3)^3}\right]$$

$$= \frac{1}{27} \times 1 - \frac{1}{27}e^{-3t} - \frac{1}{9}e^{-3t}t - \frac{1}{3}e^{-3t}\frac{t^2}{2!}$$

$$= \frac{1}{27} - \frac{e^{-3t}}{27} - \frac{e^{-3t}t}{9} - \frac{t^2e^{-3t}}{6}.$$

INVERSE LT BY PARTIAL FRACTION METHOD

Example Find $L^{-1}\left[\frac{2s+1}{(s+2)^2(s-1)^2}\right]$.

Solution. Let $\frac{2s+1}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$

$$= \frac{A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2}{(s+2)^2(s-1)^2}$$

$\therefore 2s+1 = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2.$

When $s = -2$ When $s = 1$

$9B = -3 \Rightarrow B = -\frac{1}{3}.$ $9D = 3 \Rightarrow D = \frac{1}{3}.$

Equating the coefficients of s^3 we get

$A + C = 0.$ (1)

Equating the constants we get

$$2A + B - 4C + 4D = 1$$

$$2A - \frac{1}{3} - 4C + \frac{4}{3} = 1$$

$$2A - 4C + 1 = 1$$

$$2A - 4C = 0$$

INVERSE LT BY PARTIAL FRACTION METHOD

$$A - 2C = 0. \qquad (2)$$

$$(1) - (2) \Rightarrow 3C = 0 \Rightarrow C = 0.$$

$$(1) \Rightarrow A = 0.$$

$$\therefore \frac{2s+1}{(s+2)^2(s-1)^2} = \frac{-\frac{1}{3}}{(s+2)^2} + \frac{\frac{1}{3}}{(s-1)^2}$$

Now, $L^{-1}\left[\frac{2s+1}{(s+2)^2(s-1)^2}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right]$

$$= -\frac{1}{3}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{3}e^{t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= -\frac{1}{3}e^{-2t}t + \frac{1}{3}e^{t}t = \frac{t}{3}(e^{t} - e^{-2t}).$$

INVERSE LT BY PARTIAL FRACTION METHOD

Example Find $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$.

Solution. Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$

$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

Put $s = 1 \Rightarrow 8A = 8 \Rightarrow A = 1.$

Equating the coeff. $s^2 \Rightarrow A + B = 0 \Rightarrow B = -1.$

$s = 0 \Rightarrow 5A - C = 3 \Rightarrow C = 5 - 3 = 2.$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}\right]$$

$$= e^t - L^{-1}\left[\frac{s-2}{(s+1)^2+5-1}\right]$$

$$= e^t - L^{-1}\left[\frac{s+1-3}{(s+1)^2+4}\right]$$

$$= e^t - e^{-t}L^{-1}\left[\frac{s-3}{s^2+4}\right]$$

$$= e^t - e^{-t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

$$= e^t - \frac{e^{-t}}{2}[2\cos 2t - 3\sin 2t].$$

INVERSE LT BY PARTIAL FRACTION METHOD

Example Find $L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right]$. [Dec 2013]

Solution. Let $\frac{3s^2+16s+26}{s(s^2+4s+13)} = \frac{A}{s} + \frac{Bs+c}{s^2+4s+13}$

$$3s^2+16s+26 = A(s^2+4s+13)+(Bs+c)s.$$

When $s = 0$, $13A = 26 \Rightarrow A = 2$.

Equating the coefficients of s^2

$$A + B = 3$$

$$2 + B = 3$$

$$B = 1.$$

Equating the coefficients of s

$$4A + c = 16$$

$$8 + c = 16$$

$$c = 8.$$

$$\therefore \frac{3s^2+16s+26}{s(s^2+4s+13)} = \frac{2}{s} + \frac{s+8}{s^2+4s+13}.$$

INVERSE LT BY PARTIAL FRACTION METHOD

$$L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right] = L^{-1}\left[\frac{2}{s}+\frac{s+8}{s^2+4s+13}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= 2\cdot 1 + L^{-1}\left[\frac{s+2+6}{(s+2)^2+13-4}\right]$$

$$= 2 + L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right]$$

$$= 2 + e^{-2t}L^{-1}\left[\frac{s+6}{s^2+9}\right]$$

$$= 2 + e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+9}\right) + 2L^{-1}\left(\frac{3}{s^2+9}\right)\right]$$

$$= 2 + e^{-2t}[\cos 3t + 2\sin 3t].$$

THEOREMS OF INVERSE LT

Results

1. If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then
   $L^{-1}[sF(s)] = f'(t) = \frac{d}{dt}[f(t)] = \frac{d}{dt}[L^{-1}[F(s)]]$.
   In general $L^{-1}[s^n F(s)] = f^{(n)}(t)$ if $f(0) = 0 = f'(0) = \cdots = f^{(n-1)}(0)$.
   i.e., $L^{-1}[s^n F(s)] = \frac{d^n}{dt^n}[L^{-1}[F(s)]]$.

2. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[\frac{F(s)}{s}] = \int_0^t f(t)dt = \int_0^t L^{-1}[F(s)]dt$
   Similarly, $L^{-1}[\frac{F(s)}{s^2}] = \int_0^t \int_0^t L^{-1}[F(s)]dtdt$.

3. We know that $L[tf(t)] = -\frac{d}{ds}[f(s)] = -F'(s)$.

$$L^{-1}[F'(s)] = -tf(t) = -tL^{-1}[F(s)].$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$. [May 2008]

Solution. $L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = L^{-1}\left[s\frac{1}{(s+2)^2+4}\right] = \frac{d}{dt}\left(L^{-1}\left[\frac{1}{(s+2)^2+4}\right]\right)$

$$= \frac{d}{dt}\left(e^{-2t}L^{-1}\left[\frac{1}{s^2+4}\right]\right)$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}L^{-1}\left[\frac{2}{s^2+4}\right]\right)$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin 2t\right)$$

$$= \frac{1}{2}\left(e^{-2t}2\cos 2t - 2e^{-2t}\sin 2t\right)$$

$$= e^{-2t}(\cos 2t - \sin 2t).$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{s^2}{(s-1)^4}\right]$. [Dec 2008]

Solution. $L^{-1}\left[\frac{s^2}{(s-1)^4}\right] = L^{-1}\left[s\frac{s}{(s-1)^4}\right] = \frac{d}{dt}\left(L^{-1}\left[\frac{s}{(s-1)^4}\right]\right)$

$$= \frac{d}{dt}\left(L^{-1}\left[\frac{s+1-1}{(s-1)^4}\right]\right) = \frac{d}{dt}\left(e^t L^{-1}\left[\frac{s+1}{s^4}\right]\right)$$

$$= \frac{d}{dt}\left(e^t L^{-1}\left[\frac{1}{s^3} + \frac{1}{s^4}\right]\right) = \frac{d}{dt}\left(e^t L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{1}{s^4}\right]\right)$$

$$= \frac{d}{dt}\left(e^t L^{-1}\left[\frac{t^2}{2} + \frac{t^3}{6}\right]\right) = \frac{1}{6}\frac{d}{dt}(e^t(3t^2 + t^3))$$

$$= \frac{1}{6}\left(e^t(6t + 3t^2) + e^t(3t^2 + t^3)\right) = \frac{e^t}{6}\left(6t + 6t^2 + t^3\right).$$

PROBLEMS OF INVERSE LT

Example Find $L^{-1}\left[\frac{1}{s^2(s+a)}\right]$. [Nov 2009]

Solution. $L^{-1}\left[\frac{1}{s^2(s+a)}\right] = L^{-1}\left[\frac{F(s)}{s^2}\right]$ where $F(s) = \frac{1}{s+a}, L^{-1}[F(s)] = e^{-at}$

$$= \int_0^t \int_0^t L^{-1}[F(s)]dtdt$$

$$= \int_0^t \int_0^t e^{-at}dtdt = \int_0^t \left(\frac{e^{-at}}{-a}\right)_0^t dt$$

$$= \frac{-1}{a}\int_0^t ((e^{-at}-1))dt = \frac{-1}{a}\left[\left(\frac{e^{-at}}{-a}\right)_0^t - (t)_0^t\right]$$

$$= \frac{-1}{a}\left[\frac{e^{-at}}{-a} + \frac{1}{a} - t\right]$$

$$= \frac{1}{a^2}e^{-at} - \frac{1}{a^2} + \frac{1}{a}t$$

$$= \frac{1}{a^2}\left(e^{-at} - 1 + at\right).$$

PROBLEMS OF INVERSE LT OF LOGARITHMIC FUNCTIONS

Inverse Laplace Transform of Logarithmic Functions

Worked Examples

Example Find $L^{-1}\left[\log \frac{1+s}{s^2}\right]$. [Dec 2009]

Solution. Let $F(s) = \log \frac{s+1}{s^2} = \log(s+1) - \log s^2 = \log(s+1) - 2\log s$

$$F'(S) = \frac{1}{s+1} - \frac{2}{s}$$

we know that $L[tf(t)] = -F'(s)$

$$= -\left[\frac{1}{s+1} - \frac{2}{s}\right] = \frac{2}{s} - \frac{1}{s+1}$$

$$tf(t) = L^{-1}\left[\frac{2}{s} - \frac{1}{s+1}\right] = 2L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$\therefore tf(t) = 2 - e^{-t}$$

$$f(t) = \frac{2 - e^{-t}}{t}.$$

PROBLEMS OF INVERSE LT OF LOGARITHMIC FUNCTIONS

Example . Find $L^{-1}\left[\log \frac{s^2+a^2}{s^2-b^2}\right]$. [Jun 2002]

Solution. Let $F(s) = \log \frac{s^2+a^2}{s^2-b^2} = \log(s^2+a^2) - \log(s^2-b^2)$

$$F'(S) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2-b^2}.$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2-b^2}$$

$$L^{-1}[F'(s)] = 2L^{-1}\left[\frac{s}{s^2+a^2}\right] - 2L^{-1}\left[\frac{s}{s^2-b^2}\right]$$

$$-tf(t) = 2\cos at - 2\cosh bt$$

$$f(t) = \frac{2}{t}(\cosh bt - \cos at).$$

Example Find $L^{-1}\left[\log \frac{s+1}{s-1}\right]$. [Dec 2013]

Solution. Let $F(s) = \log \frac{s+1}{s-1} = \log(s+1) - \log(s-1)$

$$F'(S) = \frac{1}{s+1} - \frac{1}{s-1}$$

We know that

$$L[tf(t)] = -F'(s) = -\frac{1}{s+1} + \frac{1}{s-1}$$

$$\therefore tf(t) = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right] = e^t - e^{-t}$$

$$\therefore f(t) = \frac{e^t - e^{-t}}{t}.$$

PROBLEMS OF INVERSE LT OF TRIGONOMETRIC FUNCTIONS

Inverse Laplace Transform of inverse Trignometric Functions

Example Find $L^{-1}\left[\tan^{-1}\frac{a}{s}\right]$.

Solution. Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right)$

$$F'(s) = \frac{1}{1+\frac{a^2}{s^2}}\left(\frac{-a}{s^2}\right) = \frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right) = -\frac{a}{s^2+a^2}.$$

We have $L[tf(t)] = -F'(s) = \frac{a^2}{s^2+a^2}$

$$tf(t) = L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$$

$$f(t) = \frac{\sin at}{t}.$$

PROBLEMS OF INVERSE LT OF TRIGONOMETRIC FUNCTIONS

Example Find $L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]$. [May 2008]

Solution. Let $F(s) = \cot^{-1}\left(\frac{2}{s+1}\right)$.

$$F'(s) = -\frac{1}{1+\frac{4}{(s+1)^2}}\left(\frac{-2}{(s+1)^2}\right) = \frac{2}{(s+1)^2+4}.$$

$$L[tf(t)] = -F'(s) = \frac{-2}{(s+1)^2+4}.$$

$$tf(t) = L^{-1}\left[-\frac{2}{(s+1)^2+4}\right] = -e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right] = -e^{-t}\sin 2t.$$

$$f(t) = -e^{-t}\frac{\sin 2t}{t}.$$

CONVOLUTION THEOREM

Inverse Laplace transform by the method of convolution

Convolution. Let $f(t)$ and $g(t)$ be two functions defined for all $t \geq 0$. The convolution of $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

Convolution theorem. If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$ then $L[f(t) * g(t)] = F(s).G(s)$. Also $L^{-1}[F(s)G(s)] = f(t) * g(t) = L^{-1}[F(s)] * L^{-1}[G(s)]$.

PROBLEMS ON CONVOLUTION THEOREM

Example Using Convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ [May 2011]

Solution. $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)} \cdot \frac{1}{(s+b)}\right]$

$$= L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt} = \int_0^t e^{-au} \cdot e^{-b(t-u)} du$$

PROBLEMS ON CONVOLUTION THEOREM

$$= \int_0^t e^{-au} \cdot e^{-bt+bu} du = \int_0^t e^{-au} \cdot e^{-bt} \cdot e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du = e^{-bt} \left[\frac{e^{(b-a)u}}{b-a} \right]_0^t$$

$$= \frac{e^{-bt}}{b-a} \left(e^{(b-a)t} - 1 \right)$$

$$= \frac{1}{b-a} \left[e^{-bt+bt-at} - e^{-bt} \right]$$

$$= \frac{1}{b-a} \left[e^{-at} - e^{-bt} \right].$$

PROBLEMS ON CONVOLUTION THEOREM

Example Using convolution theorem, evaluate $L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$.

[May 2007]

Solution. $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1}\cdot\frac{1}{s+2}$

Let $F(s) = \frac{1}{s+1}$ *and* $G(s) = \frac{1}{s+2}$.

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s+2}\right] = e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-u}e^{-2(t-u)}\,du = \int_0^t e^{-u}e^{-2t+2u}\,du$$

$$= \int_0^t e^{-2t+u}\,du = \int_0^t e^{-2t}e^{u}\,du$$

$$= e^{-2t}.(e^{u})_0^t = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}.$$

PROBLEMS ON CONVOLUTION THEOREM

Example. Using convolution theorem, evaluate $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$.

[Nov 2004]

Solution. $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = L^{-1}\left[\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$

$$= e^{-t} * \sin t = \int_0^t \sin u e^{-(t-u)} du$$

$$= \int_0^t \sin u e^{-t+u} du = \int_0^t \sin u e^{-t} e^{u} du$$

$$= e^{-t}\int_0^t e^{u} \sin u du = e^{-t}\left[\frac{e^{u}}{2}(\sin u - \cos u)\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[e^{t}(\sin t - \cos t) + 1\right] = \frac{1}{2}\left[\sin t - \cos t + e^{-t}\right].$$

PROBLEMS ON CONVOLUTION THEOREM

Example Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$ using convolution theorem. [Jun 2008]

Solution. $\frac{1}{s(s^2-a^2)} = \frac{1}{s}\cdot\frac{1}{s^2-a^2}$

Let $F(s) = \frac{1}{s}$ *and* $G(s) = \frac{1}{s^2-a^2}$

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2-a^2}\right] = 1 * \frac{1}{a}\sinh at$$

$$= \frac{1}{a}\int_0^t \sinh au.1du = \frac{1}{a}\left(\frac{\cosh au}{a}\right)_0^t = \frac{1}{a^2}\left(\cosh at - 1\right).$$

PROBLEMS ON CONVOLUTION THEOREM

Example Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem.

[Dec 2015, Dec 2014, Jun 2014, Dec 2010, May 2003]

Solution. $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)du = \int_0^t \cos au \cos(bt-bu)du$$

$$= \frac{1}{2}\int_0^t \{\cos(au+bt-bu) + \cos(au-bt+bu)\}du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{a+b+a-b}{a^2-b^2}\sin at + \frac{a-b-a-b}{a^2-b^2}\sin bt\right]$$

$$= \frac{1}{2(a^2-b^2)}\left[2a\sin at - 2b\sin bt\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

PROBLEMS ON CONVOLUTION THEOREM

Example Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.

[Jun 2012, Jun 2010, May 2008]

Solution. Let $\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$.

$$\therefore F(s) = \frac{s}{s^2+a^2}, G(s) = \frac{1}{s^2+a^2}.$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[F(s)G(s)\right] = L^{-1}F(s) * L^{-1}G(s)$$

PROBLEMS ON CONVOLUTION THEOREM

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * \left[\frac{1}{(s^2+a^2)}\right] = \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t 2\sin(at-au)\cos au du$$

$$= \frac{1}{2a}\int_0^t \{\sin(at-au+au)+\sin(at-au-au)\}du$$

$$= \frac{1}{2a}\left\{\int_0^t \sin at du + \int_0^t \sin(at-2au)du\right\}$$

$$= \frac{1}{2a}\left[\sin at(u)_0^t - \left[\frac{\cos(at-2au)}{-2a}\right]_0^t\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}(\cos(-at)-\cos at)\right]$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}0\right] = \frac{t\sin at}{2a}.$$

PROBLEMS ON CONVOLUTION THEOREM

Example Using convolution theorem, find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]$.

[Jun 2013, May 2006]

Solution. $L^{-1}\left[\frac{4}{(s^2++2s+5)^2}\right] = L^{-1}\left[\frac{4}{(s+1)^2+4)^2}\right] = e^{-t}L^{-1}\left[\frac{4}{(s^2+4)^2}\right]$ (1)

Now, $L^{-1}\left[\frac{4}{(s^2+4)^2}\right] = L^{-1}\left[\frac{2}{s^2+4}\frac{2}{s^2+4}\right] = L^{-1}\left[\frac{2}{s^2+4}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$

$$= \sin 2t * \sin 2t = \int_0^t \sin 2u \sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \left(\cos\{2u-2(t-u)\} - \cos\{2u+2t-2u\}\right)du$$

$$= \frac{1}{2}\left[\int_0^t \cos(4u-2t)du - \int_0^t \cos 2t du\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(4u-2t)}{4}\right)_0^t - \cos 2t(u)_0^t\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t\cos 2t\right] = \frac{1}{2}\left[\frac{\sin 2t}{2} - t\cos 2t\right].$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Solution of linear second order differential equations

Worked Examples

Example Solve $y'' + 5y' + 6y = 2$ given $y'(0) = 0$ and $y(0) = 0$ using Laplace transform method. [Jun 2013]

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SOLVING ODE BY USING LAPLACE TRANSFORMS

Solution. $y'' + 5y' + 6y = 2.$

Taking Laplace transform both sides we get

$$L[y''] + 5L[y'] + 6L[y] = L[2]$$

$$s^2L[y] - sy(0) - y'(0) + 5[sL[y] - y(0)] + 6L[y] = 2 \cdot L[1]$$

$$L[y][s^2 + 5s + 6] = \frac{2}{s}$$

$$L[y] = \frac{2}{s(s^2 + 5s + 6)}$$

$$= \frac{2}{s(s+2)(s+3)}$$

$$y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

Let $\frac{2}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$

$$= \frac{A(s+2)(s+3) + Bs(s+3) + Cs(s+2)}{s(s+2)(s+3)}$$

$$\therefore 2 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2).$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

When $s = 0, 6A = 2 \Rightarrow A = \frac{1}{3}$.

When $s = -2, -2B = 2 \Rightarrow B = -1$.

When $s = -3, 3C = 2 \Rightarrow C = \frac{2}{3}$.

$$\therefore \frac{2}{s(s+2)(s+3)} = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s+3}$$

$$\text{Now } y = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

$$= L^{-1}\left[\frac{1}{3}\cdot\frac{1}{s} - \frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s+3}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{3}\times 1 - e^{-2t} + \frac{2}{3}e^{-3t}$$

$$y = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}.$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Example Using Laplace transform, solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$ with $y(0) = 1 = y'(0)$. [Dec 2010, Jun 2010]

Solution. $y'' - 3y' - 4y = 2e^{-t}$.

Taking Laplace transform both sides we get

$$L[y''] - 3L[y'] - 4L[y] = 2L[e^{-t}]$$

$$s^2L[y] - sy(0) - y'(0) - 3[sL[y] - y(0)] - 4L[y] = \frac{2}{s+1}$$

$$s^2L[y] - s - 1 - 3[sL[y] - 1] - 4L[y] = \frac{2}{s+1}$$

$$L[y][s^2 - 3s - 4] - s - 1 + 3 = \frac{2}{s+1}$$

$$L[y](s-4)(s+1) - s + 2 = \frac{2}{s+1}$$

$$L[y](s-4)(s+1) = \frac{2}{s+1} + s - 2$$

$$= \frac{2 + (s+1)(s-2)}{s+1} = \frac{2 + s^2 - s - 2}{s+1} = \frac{s^2 - s}{s+1}$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$L[y] = \frac{s^2 - s}{(s+1)(s-4)(s+1)}$$

$$= \frac{s^2 - s}{(s-4)(s+1)^2}.$$

$$y = L^{-1}\left[\frac{s^2 - s}{(s-4)(s+1)^2}\right].$$

Let $\dfrac{s^2 - s}{(s-4)(s+1)^2} = \dfrac{A}{s-4} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$

$$= \frac{A(s+1)^2 + B(s-4)(s+1) + C(s-4)}{(s-4)(s+1)^2}$$

$$s^2 - s = A(s+1)^2 + B(s-4)(s+1) + C(s-4).$$

When $s = 4,\ 25A = 12 \Rightarrow A = \dfrac{12}{25}.$

When $s = -1, -5C = 2 \Rightarrow C = -\dfrac{2}{5}.$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Equating the coefficient of s^2 we get

$$A + B = 1 \Rightarrow \frac{12}{25} + B = 1 \Rightarrow B = 1 - \frac{12}{25} = \frac{13}{25}.$$

$$\therefore \frac{s^2 - s}{(s-4)(s+1)^2} = \frac{12}{25}\cdot\frac{1}{s-4} + \frac{13}{25}\cdot\frac{1}{s+1} - \frac{2}{5}\cdot\frac{1}{(s+1)^2}$$

$$\text{Now, } y = L^{-1}\left[\frac{s^2 - s}{(s-4)(s+1)^2}\right]$$

$$= L^{-1}\left[\frac{12}{25}\cdot\frac{1}{s-4} + \frac{13}{25}\cdot\frac{1}{s+1} - \frac{2}{5}\cdot\frac{1}{(s+1)^2}\right]$$

$$= \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y = \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t}.$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Example Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \sin t$, if $\frac{dy}{dt} = 0$ and $y = 2$ when $t = 0$, using Laplace transforms. [Dec 2011]

Solution. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \sin t.$

Taking Laplace transform both sides we get

$$L\left[\frac{d^2y}{dt^2}\right] + 4L\left[\frac{dy}{dt}\right] + 4L[y] = L[\sin t]$$

$$s^2L[y] - sy(0) - y'(0) + 4[sL[y] - y(0)] + 4L[y] = \frac{1}{s^2+1}$$

$$s^2L[y] - 2s + 4[sL[y] - 2] + 4L[y] = \frac{1}{s^2+1}$$

$$L[y][s^2 + 4s + 4] - 2s - 8 = \frac{1}{s^2+1}$$

$$L[y](s+2)^2 = \frac{1}{s^2+1} + 2s + 8$$

$$= \frac{1 + (2s+8)(s^2+1)}{s^2+1}$$

$$= \frac{1 + 2s^3 + 2s + 8s^2 + 8}{s^2+1}$$

$$= \frac{2s^3 + 8s^2 + 2s + 9}{s^2+1}$$

$$L[y] = \frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)}$$

$$y = L^{-1}\left[\frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)}\right].$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Let $\frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+1}$

$$= \frac{A(s+2)(s^2+1)+B(s^2+1)+(Cs+D)(s+2)^2}{(s+2)^2(s^2+1)}$$

$$\therefore 2s^3+8s^2+2s+9 = A(s+2)(s^2+1)+B(s^2+1)+(Cs+D)(s+2)^2.$$

When $s=-2$, $5B=21 \Rightarrow B=\frac{21}{5}$.

Equating the coefficients of s^3, we get

$A+C=2.$ (1)

Equating the constants we get

$2A+B+4D=9.$

$2A+\frac{21}{5}+4D=9$

$2A+4D=9-\frac{21}{5}=\frac{24}{5}$ (2)

Equating the coefficients of s^2 we get

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$2A + B + 4C + D = 8.$$

$$2A + \frac{21}{5} + 4C + D = 8$$

$$2A + 4C + D = 8 - \frac{21}{5}$$

$$2A + 4C + D = \frac{19}{5}. \quad (3)$$

From (1), $C = 2 - A$.

$$(3) \Rightarrow 2A + 4(2 - A) + D = \frac{19}{5}$$

$$2A + 8 - 4A + D = \frac{19}{5}$$

$$-2A + D = \frac{19}{5} - 8 = -\frac{21}{5}$$

$$2A - D = \frac{21}{5}. \quad (4)$$

$$(2) - (4) \Rightarrow 5D = \frac{24}{5} - \frac{21}{5} = \frac{3}{5}.$$

$$D = \frac{3}{25}.$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$(4) \Rightarrow 2A - \frac{3}{5} = \frac{21}{5}$$

$$2A = \frac{21}{5} + \frac{3}{5} = \frac{24}{5}.$$

$$A = \frac{12}{5}.$$

$$C = 2 - A = 2 - \frac{12}{5} = -\frac{2}{5}.$$

$$\therefore \frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)} = \frac{12}{5}\frac{1}{s+2} + \frac{21}{5}\cdot\frac{1}{(s+2)^2} - \frac{2}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1}$$

$$\text{Now, } y = L^{-1}\left[\frac{2s^3 + 8s^2 + 2s + 9}{(s+2)^2(s^2+1)}\right]$$

$$= L^{-1}\left[\frac{12}{5}\frac{1}{s+2} + \frac{21}{5}\cdot\frac{1}{(s+2)^2} - \frac{2}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1}\right]$$

$$= \frac{12}{5}L^{-1}\left[\frac{1}{s+2}\right] + \frac{21}{5}L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{2}{5}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{12}{5}e^{-2t} + \frac{21}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{2}{5}\cos t + \frac{3}{5}\sin t$$

$$= \frac{12}{5}e^{-2t} + \frac{21}{5}e^{-2t}t - \frac{2}{5}\cos t + \frac{3}{5}\sin t.$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Example Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = \cos 2t, y = 2$ and $\frac{dy}{dt} = 1$ where $t = 0$.

[May 2002]

Solution. The given differential equation is $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = \cos 2t.$

Taking Laplace transform on both sides, we get

$$L\left[\frac{d^2y}{dt^2}\right] + 4L\left[\frac{dy}{dt}\right] + 8L[y] = L[\cos 2t]$$

$$s^2L[y] - sy(0) - y'(0) + 4[sL(y) - y(0)] + 8L[y] = \frac{s}{s^2+4}$$

$$s^2L[y] - 2s - 1 + 4[sL(y) - 2] + 8L[y] = \frac{s}{s^2+4}$$

$$L[y][s^2 + 4s + 8] - 2s - 1 - 8 = \frac{s}{s^2+4}$$

$$L[y][s^2 + 4s + 8] = 2s + 9 + \frac{s}{s^2+4}$$

$$= \frac{(2s+9)(s^2+4)+s}{s^2+4}$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$= \frac{2s^3 + 8s + 9s^2 + 36 + s}{s^2 + 4} = \frac{2s^3 + 9s^2 + 9s + 36}{s^2 + 4}$$

$$L[y] = \frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)}$$

$$y = L^{-1}\left[\frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)}\right].$$

Let $\frac{2s^3 + 9s^2 + 9s + 36}{(s^2 + 4)(s^2 + 4s + 8)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 4s + 8}$

$$2s^3 + 9s^2 + 9s + 36 = (As + B)(s^2 + 4s + 8) + (Cs + D)(s^2 + 4)$$

Equating the coefficients of s^3 we get

$$A + C = 2. \tag{1}$$

Equating the coefficients of s^2 we get

$$4A + B + D = 9. \tag{2}$$

Equating the coefficients of s we get

$$8A + 4B + 4C = 9. \tag{3}$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

Equating the coefficients of s we get

$$8A + 4B + 4C = 9. \tag{3}$$

Equating the constants we get, $8B + 4D = 36$

$$2B + D = 9. \tag{4}$$

$$(3) \Longrightarrow 8A + 4B + 4(2 - A) = 9 \qquad \text{[using (1)]}$$

$$8A + 4B + 8 - 4A = 9$$

$$4A + 4B = 1. \tag{5}$$

$$(2) \Longrightarrow 4A + B + 9 - 2B = 9 \qquad \text{[from (4)]}$$

$$4A - B = 0$$

$$B = 4A \tag{6}$$

$$(5) \Longrightarrow 4A + 4(4A) = 1 \Longrightarrow 20A = 1 \Longrightarrow A = \frac{1}{20}.$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$(6) \Longrightarrow B = 4\frac{1}{20} = \frac{1}{5}.$$

$$(1) \Longrightarrow C = 2 - A = 2 - \frac{1}{20} = \frac{39}{20}.$$

$$(4) \Longrightarrow D = 9 - 2B = 9 - \frac{2}{5} = \frac{43}{5}.$$

$$\therefore \frac{2s^3 + 9s^2 + 9s + 36}{(s^2+4)(s^2+4s+8)} = \frac{\frac{1}{20}s + \frac{1}{5}}{s^2+4} + \frac{\frac{39}{20}s + \frac{43}{5}}{s^2+4s+8}$$

$$\text{Now, } y = L^{-1}\left[\frac{2s^3 + 9s^2 + 9s + 36}{(s^2+4)(s^2+4s+8)}\right] = L^{-1}\left[\frac{\frac{1}{20}s + \frac{1}{5}}{s^2+4}\right] + L^{-1}\left[\frac{\frac{39}{20}s + \frac{43}{5}}{s^2+4s+8}\right]$$

SOLVING ODE BY USING LAPLACE TRANSFORMS

$$= \frac{1}{20} L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{5} L^{-1}\left[\frac{1}{s^2+4}\right] + \frac{39}{20} L^{-1}\left[\frac{s}{s^2+4s+8}\right] + \frac{43}{5} L^{-1}\left[\frac{1}{s^2+4s+8}\right]$$

$$= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20} L^{-1}\left[\frac{s}{(s+2)^2+4}\right] + \frac{43}{5} L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20} e^{-2t} L^{-1}\left[\frac{s-2}{s^2+4}\right] + \frac{43}{5} e^{-2t} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20} e^{-2t}\cos 2t - \frac{39}{20} e^{-2t}\sin 2t + \frac{43}{5} e^{-2t}\sin 2t$$

$$= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t + \frac{39}{20} e^{-2t}\cos 2t + \frac{47}{20} e^{-2t}\sin 2t.$$


Thank
you