UNIT V: COMPLEX INTEGRATION

Introduction to complex integration:

Contour Integrals (complex line integral):

$Let\ f\ be\ defined\ at\ points\ of\ a\ smooth\ curve\ C$

$given\ by\ z = x(t) + iy(t), a \leq t \leq b$

$The\ \textbf{contour integral}\ of\ f\ along\ C\ is$

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\in_k)\Delta z_k\ ,$$

$where\ \ \in_k\ \ is\ the\ arbitrary\ point\ in\ the\ arc\ z_{k-1}z_k.$

Evaluation of complex integrals:

Generally, a complex integral is expressed in terms of two real integrals and evaluated.

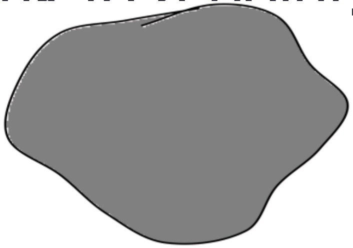
If $f(z) = u + iv$ where $z = x + iy => dz = dx + idy$

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx)$$

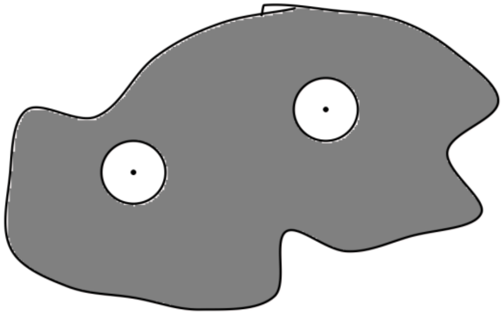
Simply connected Regions:

A region R is called simply connected, if any simple closed curve in R can be shrunk to a point.



Multiply connected Regions:

A region R which is not simply connected is called multiply connected.



Statement and application of Cauchy's integral theorem & integral formula:

Cauchy Integral Theorem:

If $f(z)$ *is analytic &* $f'(z)$ *is continuous inside & on a closed curve* C, *then* $\int_C f(z)dz = 0$.

Problems:

1. *Evaluate* $\int_C \frac{3z^2+7z-1}{z-2} dz$ *where* C *is the curve* $|z| = 1$

Solution:

Given $|z| = 1 \Rightarrow x^2 + y^2 = 1$

This is a circle with center $(0,0)$ & radius 1

Here $f(z) = 3z^2 + 7z - 1$

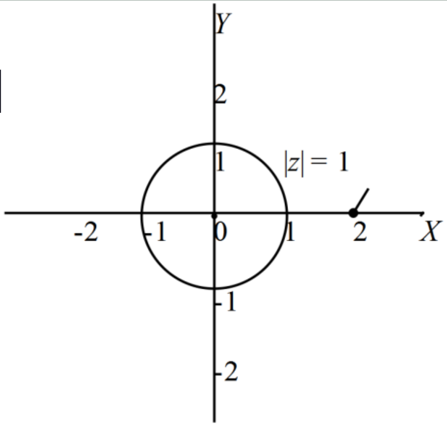
Equating denominator to zero,

$$\Rightarrow z - 2 = 0 \Rightarrow \boxed{z = 2}$$

This point lies outside the circle $|z| = 1$

$\therefore$ The function is analytic inside the circle

$\therefore$ By Cauchy Integral Theorem,



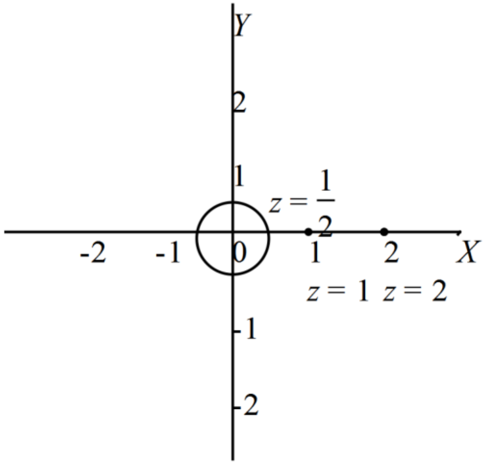
$$\boxed{\int_C \frac{3z^2 + 7z - 1}{z - 2} dz = 0}$$

2. $Evaluate \int_C \frac{2z + 5}{(z-1)(z-2)} dz \; where \; C \; is \; |z| = \frac{1}{2}$

Solution:

Given $|z| = \frac{1}{2} \Rightarrow x^2 + y^2 = \frac{1}{4}$

This is a circle with center $(0,0)$ & radius



Here $f(z) = 2z + 5$

Equating denominator to zero,

$$=> z-1=0, z-2=0$$

$$\boxed{=> z=1, z=2}$$

Both the points lies outside the circle $|z|=\frac{1}{2}$

$\therefore$ The function is analytic inside the circle

$\therefore$ By Cauchy Integral Theorem,

$$\boxed{\int_C \frac{2z+5}{(z-1)(z-2)}\,dz=0}$$

3. $Evaluate\ \int_C \frac{4z^2-6z+1}{z-4}\,dz\ , C: |z-1|=2$

Solution:

Given $|z-1|=2 => (x-1)^2+y^2=4$

This is a circle with center $(1,0)$& radius 2

Here $f(z)=4z^2-6z+1$

Equating denominator to zero,

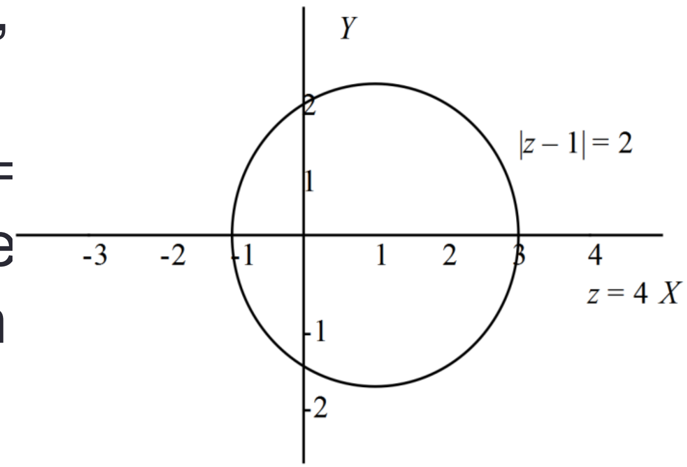
$$=> z - 4 = 0 => \boxed{z = 4}$$

This point lies outside the circle $|z - 1| =$

∴ The function is analytic inside the circle

∴ By Cauchy Integral Theorem

$$\boxed{\int_C \frac{4z^2 - 6z + 1}{z - 4} dz = 0}$$



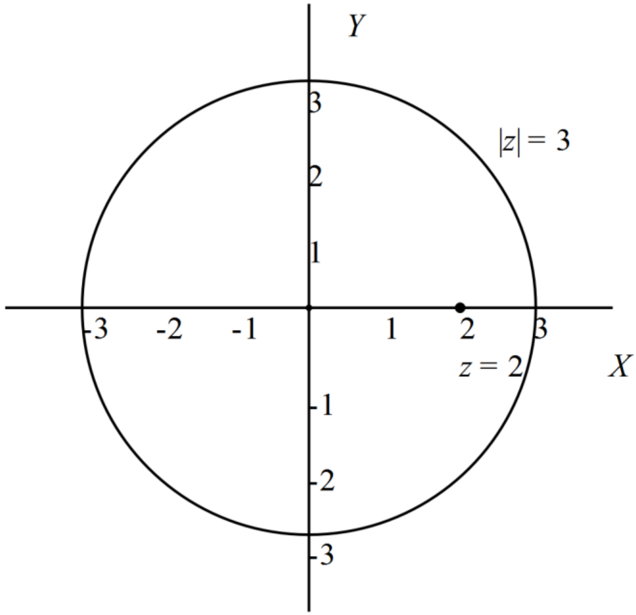
Cauchy integral formula:

If $f(z)$ is analytic inside & on a closed curve C & $'a'$ is an interior point then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz$$

Problems:

1. $Evaluate \int_C \frac{z^2+2}{z-2} dz, C: |z| = 3 \; using \; CIF$



Solution:

Given $|z| = 3 => x^2 + y^2 = 9$

This is a circle with center $(0,0)$& radius 3

Equating denominator to zero,

$=> z - 2 = 0 => \boxed{z = 2}$

This point lies inside the circle $|z| = 3$

$\therefore$ By Cauchy Integral Formula,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Here $f(z) = z^2 + 2$and $a = 2$

$\therefore f(a) = f(2) = 4 + 2 = 6$

$$\therefore \int_C \frac{z^2+2}{z-2}\,dz = 2\pi i f(2)$$

$$= 2\pi i(6) = 12\pi i$$

$$\boxed{\therefore \int_C \frac{z^2+2}{z-2}\,dz = 12\pi i}$$

2. $Evaluate\ \int_C \frac{z+1}{z^2+2z+4}\,dz, C: |z+1+i| = 2$ $using\ CIF$

Solution:

Given $|z+1+i| = 2$

$=> (x+1)^2 + (y+1)^2 = 4$

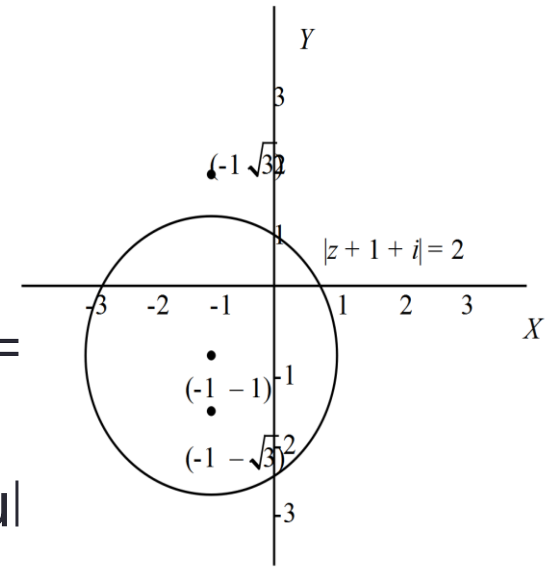
This is a circle with center $(-1,-1)$ & radius 2

Equating denominator to zero,

$=> z^2 + 2z + 4 = 0$

$$=> z = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$=> z = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$



Here $z = -1 + \sqrt{3}i$ lies outside $|z+1+i| =$
& $z = -1 - \sqrt{3}i$ lies inside $|z+1+i| = 2$

$\therefore$ By Cauchy Integral Formul

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\therefore \int_C \frac{z+1}{z^2+2z+4} dz = \int_C \frac{z+1}{[z-(-1+i\sqrt{3})][z-(-1-i\sqrt{3})]} dz$$

Here $f(z) = \dfrac{z+1}{z-(-1+i\sqrt{3})}$ and $a = -1 - \sqrt{3}i$

$$\therefore f(a) = f\left(-1-\sqrt{3}i\right)$$

$$= \frac{-1-\sqrt{3}i+1}{-1-\sqrt{3}i-\left(-1+i\sqrt{3}\right)}$$

$$= \frac{-\sqrt{3}i}{-1-\sqrt{3}i+1-\sqrt{3}i}$$

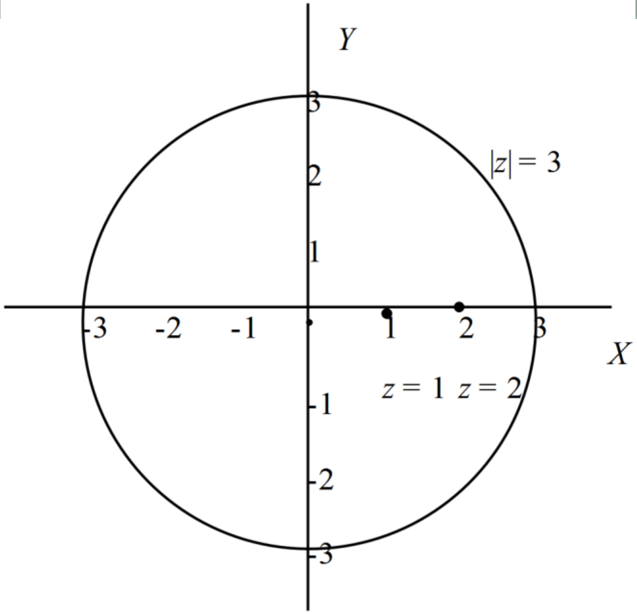
$$= \frac{-\sqrt{3}i}{-2\sqrt{3}i} = \frac{1}{2}$$

$$\therefore \int_C \frac{z+1}{z^2+2z+4}dz = 2\pi i f\left(-1-\sqrt{3}i\right)$$

$$= 2\pi i\frac{1}{2} = \pi i$$

$$\boxed{\therefore \int_C \frac{\boldsymbol{z+1}}{\boldsymbol{z^2+2z+4}}\boldsymbol{dz} = \boldsymbol{\pi i}}$$

3. $Evaluate\ \int_C \frac{\cos \pi z^2+\sin \pi z^2}{(z-1)(z-2)} dz, C: |z| = 3$ $using\ CIF$



Solution:

Given $|z| = 3 => x^2 + y^2 = 9$

This is a circle with center $(0,0)$& radius 3

Equating denominator to zero,

$=> z - 1 = 0, z - 2 = 0$

$=> z = 1, z = 2$

Both the points lies outside the circle $|z| = 3$

∴ By Cauchy Integral Formula,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$1 = A(z-2) + B(z-1) \rightarrow (1)$$

sub $z = 1$ in $(1) => 1 = A(-1) + 0$

$$\Rightarrow \boxed{A = -1}$$

sub $z = 2$ in $(1) \Rightarrow 1 = 0 + B(1) \Rightarrow \boxed{B = 1}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Now, $\displaystyle\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$

$$= -\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)} dz + \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-2)} dz$$

$$= -I_1 + I_2$$

Consider $I_1 = \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)} dz$

Here $f(z) = (\cos \pi z^2 + \sin \pi z^2)$ and $a = 1$

$$\therefore f(a) = f(1) = \cos \pi + \sin \pi = -1 + 0 = -1$$

$$\therefore I_1 = \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)} dz = 2\pi i f(1)$$

$$I_1 = -2\pi i$$

Consider $I_2 = \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-2)} dz$

Here $f(z) = (\cos \pi z^2 + \sin \pi z^2)$ and $a = 2$

$$\therefore f(a) = f(2) = \cos 4\pi + \sin 4\pi = 1 + 0 = 1$$

$$\therefore I_2 = \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-2)} dz = 2\pi i f(2) = 2\pi i$$

$$\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz = -I_1 + I_2$$

$$= -(-2\pi i) + 2\pi i = 4\pi i$$

$$\boxed{\therefore \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz = 4\pi i}$$

Taylor Series:

$If\ f(z) is\ analytic\ at\ all\ points\ inside\ C, with\ it's\ centre\ at\ a\ point' a'\ \&\ radius\ R, then\ at\ each\ point\ inside\ C,$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

Note:

Suppose if the region is given as $|z| < a,$ then change it as $\frac{|z|}{a} < 1,$ since for less than 1 the series converges fast.

We will use the following formula:

$$1. (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

$$2. (1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$$

Problems:

1\. $Expand\ f(z) = \sin z\,, using\ Taylor's\ \ series\ at\ z = \frac{\pi}{4}$

Solution:

$$\text{Given } f(z) = \sin z \quad \& \, z = \frac{\pi}{4}$$

$$\text{consider, } f(z) = \sin z \quad => f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z => f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z => f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z => f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Taylor Series is given by

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

$$f(z) = \sin z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

2. *Obtain the Taylor's series to represent the function* $\frac{1}{(z+2)(z+3)}$ *in the region* $|z| < 2$.

Solution:

Let $f(z) = \frac{1}{(z+2)(z+3)}$

$$\text{consider } \frac{1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$\Rightarrow 1 = A(z+3) + B(z+2) \rightarrow (1)$$

$$\text{sub } z = -2 \text{ in } (1) \Rightarrow 1 = A(1) + 0$$

$$\boxed{A = 1}$$

sub $z = -3$ in $(1) => 1 = 0 + B(-1)$

$$\boxed{B = -1}$$

$$\therefore \frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3}$$

Given $|z| < 2 => \frac{|z|}{2} < 1$

(Always change the condition as <1)

If $\frac{|z|}{2} < 1$, then $\frac{|z|}{3} < 1$

$$\therefore f(z) = \frac{1}{z+2} - \frac{1}{z+3}$$

$$= \frac{1}{2}\left(\frac{1}{1+\frac{z}{2}}\right) - \frac{1}{3}\left(\frac{1}{1+\frac{z}{3}}\right)$$

$$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right] - \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right]$$

$$\therefore f(z) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

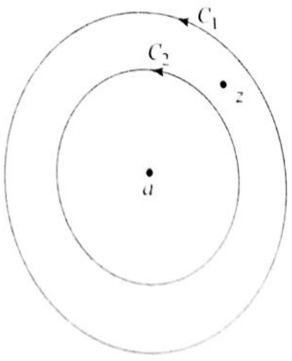
Laurent's Series:

Let C_1 *and* C_2 *be two concentric circles* $|z-a| = R_1$ & $|z-a| = R_2$ *where* $R_2 < R_1$.
Let $f(z)$ *be analytic on* C_1 & C_2 & *in the anular region* R *between them. then for any point* z *in* R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$\text{where } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$\& \; b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{1-n}}\,dz$$



Note:

The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ *, consisting of positive integral powers of* $(z-a)$ *is called the analytic part of Laurents series , while* $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ *consisting of negative integral powers of* $(z-a)$ *is called the principal part of Laurents series.*

Problems:

1. *Obtain the Laurent's series expansion of*

$\frac{1}{(z+1)(z+3)}$ *for* $(i)|z| < 1, (ii) 1 < |z| < 3, (iii)|z| > 3$

Solution:

Let $f(z) = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$

$$=> 1 = A(z+3) + B(z+1) \rightarrow (1)$$

sub $z = -1$ in $(1) => 1 = A(2) + 0$

$$\boxed{A = \frac{1}{2}}$$

sub $z = -3$ in (1) $=> 1 = 0 + B(-2)$

$$\boxed{B = -\frac{1}{2}}$$

$$\therefore f(z) = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3} \rightarrow (2)$$

$(i)|z| < 1 => \frac{|z|}{1} < 1 \; Also \; \frac{|z|}{3} < 1$

$$\therefore (2) => f(z) = \frac{1}{2}\left[\frac{1}{1+z}\right] - \frac{1}{2}.\frac{1}{3}\left[\frac{1}{1+\frac{z}{3}}\right]$$

$$= \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[(1 - z + z^2 - \cdots) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z}{3}\right)^n\right]$$

$$\therefore \boldsymbol{f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n z^n - \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^{n+1}}\right]}$$

$(ii) 1 < |z| < 3$

This condition can be spited as

$$1 < |z| \,\&\, |z| < 3$$

$$\Rightarrow \frac{1}{|z|} < 1 \,\&\, \frac{|z|}{3} < 1$$

$$\therefore (2) \Rightarrow f(z) = \frac{1}{2}\left[\frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) - \frac{1}{3}\left(\frac{1}{1+\frac{z}{3}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)\right]$$

$$\boxed{\therefore \boldsymbol{f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^{n+1} - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]}}$$

$(iii)\,|z| > 3 \Rightarrow \frac{3}{|z|} < 1.\ Also\ \frac{1}{|z|} < 1$

$$\therefore (2) \Rightarrow f(z) = \frac{1}{2}\left[\frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) - \frac{1}{z}\left(\frac{1}{1+\frac{3}{z}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) - \frac{1}{z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right) - \left(\frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^{n+1} - \sum_{n=0}^{\infty} (-1)^n 3^n \left(\frac{1}{z}\right)^{n+1}\right]$$

$$\boxed{\therefore f(z) = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (1 - 3^n) \left(\frac{1}{z}\right)^{n+1}}$$

2. *Expand* $\frac{z^2}{(z+2)(z-3)}$ *in a Laurent's series expansion if*

$$(i)|z| < 3 \ \& \ (ii) 2 < |z| < 3$$

Solution:

$$\text{Let } f(z) = \frac{z^2}{(z+2)(z-3)}$$

Here the degree of the numerator is equal to the degree of the denominator.

$\therefore$ divide numerator by denominator

$$\therefore \frac{z^2}{(z+2)(z-3)} = 1 + \frac{z+6}{(z+2)(z-3)}$$

consider $\dfrac{z+6}{(z+2)(z-3)} = \dfrac{A}{z+2} + \dfrac{B}{z-3}$

$$=> z + 6 = A(z-3) + B(z+2) \rightarrow (1)$$

sub $z = -2$ in $(1) => 4 = A(-5) + 0$

$$\boxed{A = -\frac{4}{5}}$$

sub $z = 3$ in $(1) => 9 = 0 + B(5)$

$$\boxed{B = \frac{9}{5}}$$

$$\therefore \frac{z^2}{(z+2)(z-3)} = 1 - \frac{\frac{4}{5}}{z+2} + \frac{\frac{9}{5}}{z-3} \quad \rightarrow (2)$$

$(i)|z| < 2 => \frac{|z|}{2} < 1. Also\ \frac{|z|}{3} < 1$

$$\therefore f(z) = 1 - \frac{4}{5}\left[\frac{1}{2}\left(\frac{1}{1+\frac{z}{2}}\right)\right] + \frac{9}{5}\left[-\frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right)\right]$$

$$= 1 - \frac{2}{5}\left(1+\frac{z}{2}\right)^{-1} - \frac{3}{5}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{2}{5}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right] - \frac{3}{5}\left[1 + \frac{z}{3} + \frac{z^2}{9} - \cdots\right]$$

$$= 1 - \frac{1}{5}\left[2\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - 3\sum_{n=0}^{\infty}\frac{z^n}{3^n}\right]$$

$$= 1 - \frac{1}{5}\sum_{n=0}^{\infty}\left((-1)^n\frac{2}{2^n} - \frac{3}{3^n}\right)z^n$$

$$\boxed{\therefore \boldsymbol{f(z) = 1 - \frac{1}{5}\sum_{n=0}^{\infty}\left((-1)^n\frac{1}{2^{n-1}} - \frac{1}{3^{n-1}}\right)z^n}}$$

$(ii) 2 < |z| < 3 \Rightarrow 2 < |z| \ \& \ |z| < 3$

$$\frac{2}{|z|} < 1 \;\; \& \;\; \frac{|z|}{3} < 1$$

$$\therefore \; f(z) = 1 - \frac{4}{5}\left[\frac{1}{z}\left(\frac{1}{1+\frac{2}{z}}\right)\right] + \frac{9}{5}\left[-\frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right)\right]$$

$$= 1 - \frac{4}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{3}{5}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{4}{5z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \cdots\right] - \frac{3}{5}\left[1 + \frac{z}{3} + \frac{z^2}{9} - \cdots\right]$$

$$= 1 - \frac{1}{5}\left\{\frac{4}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \cdots\right] - 3\left[1 + \frac{z}{3} + \frac{z^2}{9} - \cdots\right]\right\}$$

$$\boxed{\therefore \; \boldsymbol{f(z) = 1 - \frac{1}{5}\left[\frac{4}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n} - 3\sum_{n=0}^{\infty}\frac{z^n}{3^n}\right]}}$$

$3. \textit{Find the Laurent's series expansion for}$
$f(z) = \frac{7z-2}{(z+1)(z)(z-2)}$ *in the region* $1 < |z+1| < 3.$

Solution:

consider

$$f(z) = \frac{7z-2}{(z+1)(z)(z-2)} = \frac{A}{z+1} + \frac{B}{z} + \frac{C}{z-2}$$

$$= \frac{Az(z-2) + B(z+1)(z-2) + Cz(z+1)}{(z+1)(z)(z-2)}$$

$$7z - 2 = Az(z-2) + B(z+1)(z-2) + Cz(z+1) \rightarrow (1)$$

sub $z = -1$ in (1)

$$=> -9 = A(-1)(-3) + 0 + 0$$

$$\boxed{A = -3}$$

sub $z = 2$ in (1)

$$=> 12 = 0 + 0 + C(2)(3)$$

$$\boxed{C = 2}$$

sub $z = 0$ in (1)

$$=> \ -2 = 0 + B(1)(-2) + 0$$

$$\boxed{B = 1}$$

$$\therefore f(z) = -\frac{3}{z+1} + \frac{1}{z} + \frac{2}{z-2}$$

Given $1 < |z+1| < 3$

Let $z + 1 = u$, then $1 < |u| < 3$

$$=> 1 < |u| \,\&\, |u| < 3$$

$$=> \frac{1}{|u|} < 1 \,\&\, \frac{|u|}{3} < 1$$

$$\therefore f(u-1) = -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$$= -\frac{3}{u} + \frac{1}{u}\left[\frac{1}{\left(1 - \frac{1}{u}\right)}\right] - \frac{2}{3}\left[\frac{1}{1 - \frac{u}{3}}\right]$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{9} + \cdots\right]$$

$$= -\frac{2}{u} + \sum_{n=2}^{\infty} \frac{1}{u^n} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n$$

$$\therefore \boldsymbol{f(z) = -\frac{2}{z+1} + \sum_{n=2}^{\infty} \frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n}$$

Singular points:

A point $z = a$ *at which a function* $f(z)$ *fails to be analytic is called a singular point (or) singularity of* $f(z)$.

$$Eg: f(z) = \frac{1}{z-2} \Rightarrow z = 2 \text{ is a singular point.}$$

Classification of singularity:

Isolated singularities:

$$A\ point\ z = a\ is\ said\ to\ be\ isolated\ singularity\ \ of\ f(z) if$$

$$(i) z = a\ should\ be\ a\ singular\ point$$

$$(ii) the\ neighbourhood\ of\ z = a\ should\ not\ contain\ any\ other\ singular\ point.$$

$Eg: f(z) = \frac{1}{z(z+2)}$

$$=> z = 0, z = -2\ are\ isolated\ singular\ points.$$

Poles:

$An\ isolated\ singularity\ z = a\ is\ \ called\ \ a\ pole.$

$A\ pole\ of\ order\ one\ is\ called\ a\ simple\ pole.$

$$Eg: f(z) = \frac{1}{(z-4)^2(z-3)^3(z-1)}$$

$$Here\ z = 1\ is\ a\ simple\ pole.$$

$$z = 3\ is\ a\ pole\ of\ order\ 3$$

$$z = 4\ is\ a\ pole\ of\ order\ 2.$$

Removable singularities:

$$A\ singular\ point\ z = a\ is\ called\ a\ removable\ singularity\ of\ f(z)\ if\ \lim_{z\to a} f(z)\ exists\ finitely.$$

$$Eg: f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$$here\ z = 0\ is\ the\ singular\ point.$$

$$\therefore \lim_{z\to 0} f(z) = 1 - 0 - 0 - \cdots = 1(finite)$$

$$\therefore z = 0\ is\ a\ removable\ singularity.$$

Essential singularities:

$$If\ f(z)\ contains\ infinite\ number\ of\ negative\ powers\ terms\ then\ z = a\ is\ known\ as\ essential\ singularity.$$

$$Eg: f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \cdots$$

$$Here\ z = 0\ is\ the\ singular\ point.$$

$also\ f(z) contains\ infinite\ number\ of\ negative\ terms.$

$$\therefore z = 0\ is\ a\ essential\ singularity.$$

Problems:

1. $what\ is\ the\ nature\ of\ the\ singularity\ at\ z = 0\ of$

$f(z) = \frac{\sin z - z}{z^3}$

Solution:

$$Given\ f(z) = \frac{\sin z - z}{z^3}$$

$Here\ z = 0\ is\ the\ singular\ point.$

$$\therefore f(z) = \frac{1}{z^3}\left\{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] - z\right\}$$

$$= \frac{1}{z^3}\left[-\frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = -\frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

$\lim_{z \to 0} f(z) = -\frac{1}{3!} + 0 + 0 + \cdots = -\frac{1}{3!}(finite)$

$$\therefore z = 0\ is\ a\ removable\ singularity.$$

$2. \, Find \; singularity \; point \; of \; f(z) = \sin\left(\frac{1}{z-a}\right)$

Solution:

$$given \; f(z) = \sin\left(\frac{1}{z-a}\right)$$

$$= \frac{1}{z-a} - \frac{1}{3!}\left(\frac{1}{z-a}\right)^3 + \frac{1}{5!}\left(\frac{1}{z-a}\right)^5 - \cdots$$

$Here \; z = a \; is \; the \; singular \; point.$

$also \; f(z) contains \; infinite \; number \; of \; negative \; terms.$

$\therefore \boldsymbol{z = a \; is \; a \; essential \; singularity}.$

$3. \, Find \; the \; pole \; of \; f(z) = \frac{1}{(z^2+a^2)^2}$

Solution:

$$Given \;\; f(z) = \frac{1}{(z^2+a^2)^2}$$

$$=> \; z^2 + a^2 = 0 => z = \pm ia$$

$$\therefore \boldsymbol{z = \pm ia \; are \; poles \; of \; order \; 2}.$$

Residue:

$$\text{The coefficient of } \frac{1}{z-a} \text{ in the expansion of } f(z)$$

about the singular point $z = a$ *is defined as residue of* $f(z)$ *at* $z = a$.

Evaluation of Residue:

Residue at a pole of order m *is given by*

$$R[z = a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

→ (I)

Note:

Residue at a pole of order 1 *(simple pole) is given by*

$$R[z = a] = \frac{1}{(1-1)!} \lim_{z \to a} \frac{d^{1-1}}{dz^{1-1}} [(z-a) f(z)]$$

$$R[z = a] = \lim_{z \to a} (z-a) f(z)$$ → (II)

Problems:

1. $Calculate\ the\ residue\ of\ f(z) = \frac{1-e^{2z}}{z^3}$

Solution:

Here $z = 0$ is a pole of order 3

$$\therefore \text{(I)} => R[z=0] = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^{3-1}}{dz^{3-1}} \left[(z-0)^3 \frac{1-e^{2z}}{z^3} \right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} \left[z^3 \frac{1-e^{2z}}{z^3} \right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} [1-e^{2z}]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d}{dz} [0-2e^{2z}]$$

$$= \frac{1}{2!} \lim_{z\to 0} [-4\, e^{2z}]$$

$$= \frac{1}{2!} [-4e^0] = -2$$

$$\therefore \boldsymbol{R[z=0]=-2}$$

2. $Test\ the\ singularity\ of\ \frac{1}{z^2+1}\ \&\ hence\ find\ the\ residues.$

Solution:

$$Let\ f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$$

$\therefore z=i\ \&\ z=-i$ are simple pole

(II) $=> R[z=i]=\lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)}$

$$=\lim_{z\to i}\frac{1}{(z+i)}=\frac{1}{2i}$$

$$\therefore \boldsymbol{R[z=i]=\frac{1}{2i}}$$

(II) $=> R[z=-i]=\lim_{z\to -i}(z+i)\frac{1}{(z+i)(z-i)}$

$$= \lim_{z \to -i} \frac{1}{(z-i)} = -\frac{1}{2i}$$

$$\boxed{\therefore \boldsymbol{R[z=-i] = -\frac{1}{2i}}}$$

Cauchy Residue theorem:

If $f(z)$ is analytic at all points inside & on a simple closed curve C, except at a finite number of poles $z_1, z_2, \dots, z_n$ within C, then

$$\int_C f(z)dz = 2\pi i[\text{sum of residues of } f(z) \text{ at } z_1, z_2, \dots, z_n]$$

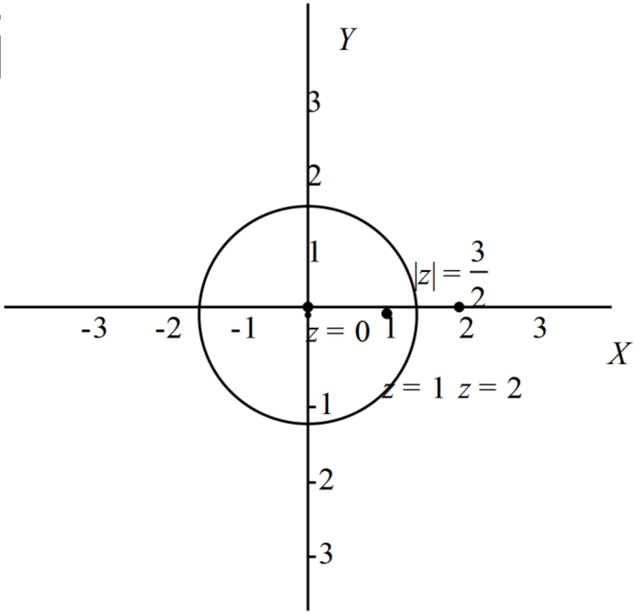
Problems:

1. $Evaluate \int_C \frac{4-3z}{z(z-1)(z-2)} dz, C: |z| = \frac{3}{2}.$

Solution:

Given $f(z) = \dfrac{4-3z}{z(z-1)(z-2)}$

$$\&|z| = \frac{3}{2} => x^2 + y^2 = \frac{9}{4}$$



This is a circle with center $(0,0)$& radius $\frac{3}{2}$

Equating denominator to zero , we get

$z = 0, z - 1 = 0 \; \& \; z - 2 = 0$

$=> \boxed{z = 0, z = 1 \; \& \; z = 2}$

Here $z = 0$ is a pole of order 1

$z = 1$ is a pole of order 1

$z = 2$ is a pole of order 1

Also $z = 0 \; \& \; z = 1$are poles which lies inside C

$\& \; z = 2$ is a pole which lie outside C

To find residue at $z = 0$:

$$R[z = 0] = \lim_{z \to 0} (z - 0) \frac{4 - 3z}{z(z - 1)(z - 2)}$$

$$= \lim_{z \to 0} \frac{4 - 3z}{(z - 1)(z - 2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

$$\boxed{\therefore \boldsymbol{R[z=0]=2}}$$

To find residue at $z = 1$:

$$R[z=1] = \lim_{z \to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 1}\frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{1(-1)} = -1$$

$$\boxed{\therefore \boldsymbol{R[z=1]=-1}}$$

∴ By Cauchy Residue Theorem,

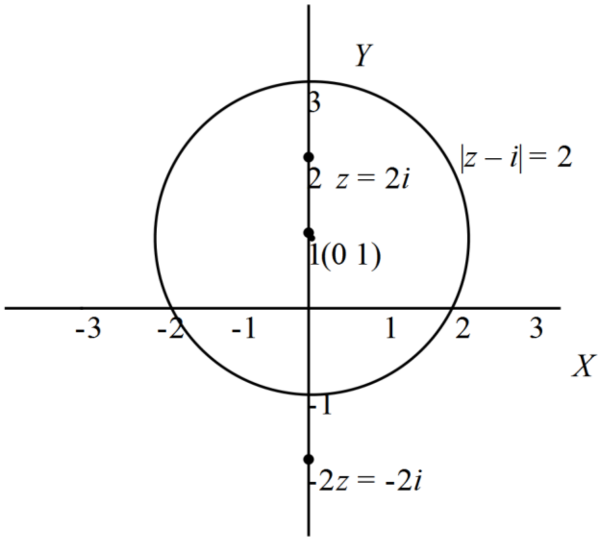
$$\int_C f(z)dz = 2\pi i[sum\ of\ residues]$$

$$= 2\pi i[2-1] = 2\pi i$$

$$\therefore \boxed{\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i}$$

2. *Evaluate* $\int_C \frac{dz}{(z^2+4)^2}, C: |z-i| = 2$ *using RT*.

Solution:



Given $f(z) = \dfrac{1}{(z^2+4)^2}$

& $|z-i| = 2 \Rightarrow x^2 + (y-1)^2 = 4$

This is a circle with center $(0,1)$ & radius 2

Equating denominator to zero , we get

$z^2 + 4 = 0 \Rightarrow \boxed{z = \pm 2i}$

Here $z = \pm 2i$ is a pole of order 2

Also $z = 2i$ is a pole which lie inside C

& $z = -2i$ is a pole which lie outside C

To find residue at $z = 2i$:

$$R[z = 2i] = \frac{1}{1!} \lim_{z \to 2i} \frac{d}{dz}\left[(z - 2i)^2 \frac{1}{(z - 2i)^2 (z + 2i)^2}\right]$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left[\frac{1}{(z + 2i)^2}\right]$$

$$= \lim_{z \to 2i} \left[\frac{-2}{(z + 2i)^3}\right] = -\frac{2}{(4i)^3}$$

$$= -\frac{2}{64i^3} = \frac{1}{32i} \quad (since\ i^2 = -1)$$

$$\boxed{\therefore\ R[z = 2i] = \frac{1}{32i}}$$

∴ By Cauchy Residue Theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{32i}\right) = \frac{\pi}{16}$$

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

Evaluation of real definite integrals by contour integration:

Type I:

Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$

$where\ f(\cos\theta - \sin\theta) is\ a\ rational\ function\ of\ \cos\theta\ \&\ \sin\theta$.

In this type of integral, put $\boxed{z = e^{i\theta}}$

then $dz = ie^{i\theta} d\theta \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

we know that,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \ \&\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\Rightarrow \boxed{\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]} \ \&\ \boxed{\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]}$$

$$\therefore \int_0^{2\pi} f(\cos\theta - \sin\theta)d\theta = \int_C f\left(\frac{1}{2}\left[z+\frac{1}{z}\right], \frac{1}{2i}\left[z-\frac{1}{z}\right]\right)\frac{dz}{iz}$$

where C is unit circle $|z| = 1$

$$= \frac{1}{i}\int_C \varphi(z)dz$$ where $\varphi(z)$ is a rational function of z.

Hence by Residue theorem,

$$\int_C \varphi(z)dz = 2\pi i[sum\ of\ residues\ of\ \varphi(z)\ at\ it'spoles\ inside\ C]$$

Problems:

1. $Evaluate\ \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

Solution:

Transforming the variable θ in terms of z

Put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

Also $\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \int_C \frac{dz/iz}{5 + 4\left[\frac{1}{2}\left[\frac{z^2+1}{z}\right]\right]}$$

where $C: |z| = 1$

$$= \int_C \frac{dz/iz}{\left(\frac{5z + 2z^2 + 2}{z}\right)}$$

$$= \frac{1}{i}\int_C \frac{dz}{2z^2 + 5z + 2}$$

$$= \frac{1}{2i}\int_C \frac{dz}{z^2 + \frac{5}{2}z + 1}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{1}{2i}\int_C f(z)dz$$

$$\text{where } f(z) = \frac{1}{z^2 + \frac{5}{2}z + 1}$$

To find poles of $f(z)$:

Equating denominator to zero , we get

$$z^2 + \frac{5}{2}z + 1 = 0$$

$$=> z = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{-\frac{5}{2} \pm \sqrt{\frac{25-16}{4}}}{2}$$

$$= \frac{-\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$

$$= \frac{-\frac{5}{2}+\frac{3}{2}}{2}, \frac{-\frac{5}{2}-\frac{3}{2}}{2}$$

$$\boxed{z = -\frac{1}{2}, -2}$$

$$\boxed{\therefore f(z) = \frac{1}{(z+1/2)(z+2)}}$$

$$Here\ z = -\frac{1}{2}\ lies\ inside\ C(order\ 1)$$

$$\&\ z = -2\ lies\ outside\ C.$$

To find residue at $\boldsymbol{z = -\frac{1}{2}}$:

$$R[z = -1/2] = \lim_{z \to -1/2} (z+1/2) \frac{1}{(z+1/2)(z+2)}$$

$$= \lim_{z \to -1/2} \frac{1}{(z+2)} = \frac{1}{\left(-\frac{1}{2}+2\right)} = \frac{2}{3}$$

$$\therefore R[z = -1/2] = \frac{2}{3}$$

Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

$$= 2\pi i\left[\frac{2}{3}\right]$$

$$\therefore \int_C f(z)dz = \frac{4\pi i}{3}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{1}{2i}\int_C f(z)dz = \frac{1}{2i}\left[\frac{4\pi i}{3}\right] = \frac{2\pi}{3}$$

$$\therefore \int_0^{2\pi} \frac{\boldsymbol{d\theta}}{\boldsymbol{5 + 4\cos\theta}} = \frac{\boldsymbol{2\pi}}{\boldsymbol{3}}$$

2. $Evaluate\ \int_C \frac{d\theta}{1-2a\sin\theta+a^2}\ , 0 < a < 1$

Solution:

Transforming the variable θ in terms of z

Put $z = e^{i\theta} => d\theta = \frac{dz}{iz}$

Also $\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right] = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

$$\therefore \int_C \frac{d\theta}{1-2a\sin\theta+a^2} = \int_C \frac{dz/iz}{1-2a\left[\frac{1}{2i}\left[\frac{z^2-1}{z}\right]\right]+a^2}$$

where $C: |z| = 1$

$$= \int_C \frac{dz/iz}{1-\frac{a}{iz}[z^2-1]+a^2}$$

$$= \int_C \frac{dz/iz}{(iz - az^2 + a + a^2 iz)/iz}$$

$$= \int_C \frac{dz}{-az^2 + i(1 + a^2)z + a}$$

$$= -\frac{1}{a} \int_C \frac{dz}{z^2 - i\left(\frac{1 + a^2}{a}\right) z - 1}$$

$$\therefore \int_C \frac{d\theta}{1 - 2a \sin\theta + a^2} = -\frac{1}{a} \int_C f(z) dz$$

where $f(z) = \dfrac{1}{z^2 - i\left(\frac{1 + a^2}{a}\right) z - 1}$

To find poles of $f(z)$:

Equating denominator to zero , we get

$$z^2 - i\left(\frac{1+a^2}{a}\right)z - 1 = 0$$

$$z = \frac{i\left(\frac{1+a^2}{a}\right) \pm \sqrt{\left(i\left(\frac{1+a^2}{a}\right)\right)^2 - 4(1)(-1)}}{2}$$

$$= \frac{i\left(\frac{1+a^2}{a}\right) \pm \sqrt{-\left(\frac{1+a^2}{a}\right)^2 + 4}}{2}$$

$$= \frac{i\left(\frac{1+a^2}{a}\right) \pm \sqrt{\frac{-1-a^4-2a^2+4a^2}{a^2}}}{2}$$

$$= \frac{i\left(\frac{1+a^2}{a}\right) \pm \sqrt{\frac{-1-a^4+2a^2}{a^2}}}{2}$$

$$= \frac{i\left(\frac{1+a^2}{a}\right) \pm \frac{i}{a}\sqrt{(1-a^2)^2}}{2}$$

$$= \frac{i\left(\frac{1+a^2}{a}\right) \pm i\left(\frac{1-a^2}{a}\right)}{2}$$

$$=> z = \frac{i}{2a}[(1+a^2) \pm (1-a^2)]$$

$$=> z = \frac{i}{2a}[(1+a^2) + (1-a^2)],$$

$$z = \frac{i}{2a}[(1+a^2) - (1-a^2)]$$

$$=> z = \frac{i}{2a}[2], z = \frac{i}{2a}[2a^2]$$

$$=> \boxed{z = \frac{i}{a}\ , z = ia}$$

$$\boxed{\therefore f(z) = \frac{1}{(z - ia)\left(z - \frac{i}{a}\right)}}$$

$$Here\ z = \frac{i}{a}\ lies\ outside\ C\ (since\ a < 1 => \frac{1}{a} > 1)$$

$$\&\ z = ia\ lies\ inside\ C\ (order\ 1)(since\ a < 1)$$

To find residue at $z = ia$:

$$R[z = ia] = \lim_{z \to ia} (z - ia) \frac{1}{(z - ia)\left(z - \frac{i}{a}\right)}$$

$$= \lim_{z \to ia} \frac{1}{\left(z - \frac{i}{a}\right)} = \frac{1}{ia - \frac{i}{a}}$$

$$\boxed{\therefore R[z = ia] = \frac{a}{i(a^2 - 1)}}$$

Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

$$= 2\pi i\left[\frac{a}{i(a^2-1)}\right]$$

$$\boxed{\int_C f(z)dz = \frac{2\pi a}{(a^2-1)}}$$

$$\therefore \int_C \frac{d\theta}{1-2a\sin\theta+a^2} = -\frac{1}{a}\int_C f(z)dz$$

$$= -\frac{1}{a}\left[\frac{2\pi a}{(a^2-1)}\right] = \frac{2\pi}{(1-a^2)}$$

$$\therefore \boxed{\int_C \frac{\boldsymbol{d\theta}}{\mathbf{1-2}\boldsymbol{a}\,\mathbf{sin}\,\boldsymbol{\theta}+\boldsymbol{a}^{\mathbf{2}}} = \frac{\mathbf{2}\boldsymbol{\pi}}{(\mathbf{1}-\boldsymbol{a}^{\mathbf{2}})}}$$

Type II :

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

where $P(x)$ & $Q(x)$ are polynomials in x

This integral converges(exists) if, Degree of $Q(x)$ is at least two greater than the degree of $P(x)$ & $Q(x)$ has no real roots.

To evaluate this integral, we consider the integral

$\int_C \frac{P(z)}{Q(z)} dz$, where C is the closed contour consisting

of the real axis from $-R$ to R & the semicircle

$\Gamma: |z| = R$ in the upper half of the complex plane.

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz \rightarrow (1)$$

where $f(z) = \frac{P(z)}{Q(z)}$

All the positive poles will lie inside & negative poles lie outside.
Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

letting $R \to \infty$, $\int_\Gamma f(z)dz = 0$(by Cauchy's Lemma)

$$\therefore (1) => \int_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz$$

Note: Cauchy's Lemma:

If $f(z)$ *is a continous function such that* $|zf(z)| \to 0$ *uniformly as* $|z| \to \infty$ *on* Γ, *then* $\int_\Gamma f(z)dz \to 0$ *as* $R \to \infty$, *where* Γ *is a semicircle* $|z| = R$ *above the real axis*.

Problems:

1. $Evaluate \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Solution:

$Let\ us\ consider$

$$\int_C f(z)dz = \int_C \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

where C is the closed contour consisting of the real axis from $-R$ to R & the semicircle $\Gamma: |z| = R$ in the upper half of the complex plane

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz \rightarrow (1)$$

To find $\int_C f(z)dz$:

$$\text{Consider,} f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

To find poles of $f(z)$:

Equating denominator to zero , we get

$$z^2 + a^2 = 0 \;\&\; z^2 + b^2 = 0$$

$$\Rightarrow z^2 = -a^2 \,, \qquad z^2 = -b^2$$

$$\Rightarrow \boxed{z = \pm ai} \,, \qquad \boxed{z = \pm bi}$$

The poles $z = ai$ &$z = bi$ lies inside C

& the poles $z = -ai$ & $z = -bi$ lies outside C

$$\therefore f(z) = \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

To find residue at $z = ia$:

$$R[z = ia] = \lim_{z \to ia} (z - ia) \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

$$= \lim_{z \to ia} \frac{1}{(z+ia)(z+ib)(z-ib)}$$

$$= \frac{1}{(ia+ia)(ia+ib)(ia-ib)} = \frac{1}{2ai(-a^2+b^2)}$$

$$\boxed{R[z = ia] = \frac{1}{2ai(-a^2 + b^2)}}$$

To find residue at $z = ib$:

$$R[z = ib] = \lim_{z \to ib} (z - ib) \frac{1}{(z + ia)(z - ia)(z + ib)(z - ib)}$$

$$= \lim_{z \to ib} \frac{1}{(z + ia)(z - ia)(z + ib)}$$

$$= \frac{1}{(ib + ia)(ib - ia)(ib + ib)} = \frac{1}{2bi(-b^2 + a^2)}$$

$$\boxed{R[z = ib] = \frac{1}{2bi(-b^2 + a^2)}}$$

Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

$$= 2\pi i\left[\frac{1}{2ai(-a^2+b^2)} + \frac{1}{2bi(-b^2+a^2)}\right]$$

$$= \pi\left[\frac{-1}{a(a^2-b^2)} + \frac{1}{b(a^2-b^2)}\right]$$

$$= \pi\left[\frac{-b+a}{ab(a^2-b^2)}\right] = \pi\left[\frac{-b+a}{ab(a+b)(a-b)}\right]$$

$$\boxed{\therefore \int_C f(z)dz = \pi\left[\frac{1}{ab(a+b)}\right]}$$

consider

$$(1) \Rightarrow \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz$$

$$\Rightarrow \pi\left[\frac{1}{ab(a+b)}\right] = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz$$

letting $R \to \infty, \int_{\Gamma} f(z)dz = 0$(by Cauchy's Lemma)

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi \left[\frac{1}{ab(a+b)}\right]$$

(since on real axis $z = x => dz = dx$)

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \pi \left[\frac{1}{ab(a+b)}\right]$$

2. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$

Solution:

Let us consider

$$\int_C f(z)dz = \int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$$

where C is the closed contour consisting of the real axis from $-R$ to R & the semicircle Γ: $|z| = R$ in the upper half of the complex plane

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz \rightarrow (1)$$

To find $\int_C f(z)dz$:

$$\text{Consider, } f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

To find poles of $f(z)$:

Equating denominator to zero , we get

$$=> z^4 + 10z^2 + 9 = 0$$

$$z^4 + z^2 + 9z^2 + 9 = 0$$

$$z^2(z^2+1) + 9(z^2+1) = 0$$

$$(z^2+1)(z^2+9) = 0$$

$$=> z^2 + 1 = 0 \;\&\; z^2 + 9 = 0$$

$$=> \boxed{z = \pm i}, \boxed{z = \pm 3i}$$

The poles $z = i$ &$z = 3i$ lies inside C

& the poles $z = -i$ & $z = -3i$ lies outside C

$$\therefore f(z) = \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)(z-3i)}$$

To find residue at $z = i$:

$$R[z = i] = \lim_{z \to i}(z - i)\frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \to i} \frac{z^2 - z + 2}{(z+i)(z+3i)(z-3i)}$$

$$= \frac{-1 - i + 2}{(2i)(4i)(-2i)}$$

$$= \frac{1-i}{-16i^3} = \frac{1-i}{16i} \quad \text{(since } i^3 = -i\text{)}$$

$$\boxed{\therefore R[z = i] = \frac{1-i}{16i}}$$

To find residue at $z = 3i$:

$$R[z = 3i] = \lim_{z \to 3i} (z - 3i) \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \to 3i} \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)}$$

$$= \frac{-9 - 3i + 2}{(4i)(2i)(6i)}$$

$$= \frac{-7-3i}{48i^3} = \frac{-7-3i}{-48i} \quad \text{(since } i^3 = -i)$$

$$\boxed{\therefore R[z = 3i] = \frac{7+3i}{48i}}$$

Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

$$= 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= 2\pi i\frac{1}{16i}\left[1 - i + \frac{7+3i}{3}\right]$$

$$= \frac{\pi}{8}\left[1 - i + \frac{7}{3} + i\right] = \frac{\pi}{8}\left[\frac{10}{3}\right] = \frac{5}{12}\pi$$

$$\boxed{\therefore \int_C f(z)dz = \frac{5}{12}\pi}$$

consider (1) $\Rightarrow \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz$

$$\Rightarrow \frac{5}{12}\pi = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz$$

letting $R \to \infty$, $\int_\Gamma f(z)dz = 0$

(by Cauchy's Lemma)

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \frac{5}{12}\pi$$

(since on real axis $z = x \Rightarrow dz = dx$)

$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5}{12}\pi}$$

Type III :

Integrals of the form $\displaystyle\int_{-\infty}^{\infty} \frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}} \cos \boldsymbol{mx\, dx}\ \ (\boldsymbol{or})$

$\int_{-\infty}^{\infty} \frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}} \sin \boldsymbol{mx\, dx}$ where $P(x)$ & $Q(x)$ are polynomials in x

This integral converges(exists) if, $(i) m > 0$

(ii)Degree of $Q(x)$ is at least two greater than the degree of $P(x)$ & $Q(x)$ has no real roots.

To evaluate this integral, we consider the integral

$\displaystyle\int_C \frac{P(z)}{Q(z)} RPe^{imz} dz\ (or) \int_C \frac{P(z)}{Q(z)} IPe^{imz} dz$ where C is the closed contour consisting of the real axis from $-R$ to R & the semicircle Γ: $|z| = R$ in the upper half of the complex plane.

Then proceed as in Type II

Problem:

$Evaluate\ \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} dx$

Solution:

$Let\ us\ consider$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} dx = \int_{-\infty}^{\infty} \frac{RP\ e^{imx}}{x^2+a^2} dx = RP \int_{-\infty}^{\infty} \frac{e^{imz}}{z^2+a^2} dz$$

$$consider \int_C f(z)dz = \int_C \frac{e^{imz}}{z^2+a^2} dz$$

where C is the closed contour consisting of the real axis from $-R$ to R & the semicircle $\Gamma: |z| = R$ in the upper half of the complex plane

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{\Gamma} f(z)dz \rightarrow (1)$$

To find $\int_C f(z)dz$:

$$\text{Consider, } f(z) = \frac{e^{imz}}{z^2 + a^2}$$

To find poles of $f(z)$:

Equating denominator to zero , we get

$$z^2 + a^2 = 0 => z = \pm ia$$

$$Here\ z = ia\ lie\ inside\ C$$

$$\&\ z = -ia\ lie\ outside\ C$$

$$\therefore f(z) = \frac{e^{imz}}{(z + ia)(z - ia)}$$

To find residue at $z = ai$:

$$R[z = ai] = \lim_{z \to ai} (z - ai) \frac{e^{imz}}{(z + ia)(z - ia)}$$

$$= \lim_{z \to ai} \frac{e^{imz}}{(z + ia)} = \frac{e^{im(ia)}}{(ia + ia)}$$

$$\therefore R[z = ai] = \frac{e^{-ma}}{2ia}$$

Hence by Residue theorem,

$$\int_C f(z)dz = 2\pi i[sum\ of\ residues\ of\ f(z)]$$

$$= 2\pi i\left[\frac{e^{-ma}}{2ia}\right] = \frac{\pi e^{-ma}}{a}$$

$$\therefore \int_C f(z)dz = \frac{\pi e^{-ma}}{a}$$

consider

$$(1) => \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{\Gamma} f(z)dz$$

$$=> \frac{\pi e^{-ma}}{a} = \int_{-R}^{R} f(z)dz + \int_{\Gamma} f(z)dz$$

letting $R \to \infty, \int_{\Gamma} f(z)dz = 0$ (by Cauchy's Lemma)

$$\therefore \int_{-\infty}^{\infty} f(x)dx = RP\left[\frac{\pi e^{-ma}}{a}\right] = \frac{\pi e^{-ma}}{a}$$

(since on real axis $z = x => dz = dx$)

$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi e^{-ma}}{a}}$$