UNIT - III

Analytic functions

Complex variable:

A complex variable is $x+iy$ and it is denoted by z. i.e. $z = x+iy$ where $i = \sqrt{-1}$

The complex constant is denoted by $a+ib$.

Functions of a complex variable:

$w = u(x,y) + iv(x,y)$ is a function of the complex variable $z = x+iy$.

i.e., $w = f(z) = u(x,y) + i\,v(x,y)$

where $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part of the complex function $f(z)$.

In general we can write $f(z) = u+iv$

Limit of a function:

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then the limit of $f(z)$ as z approaches z_0 is w_0.

$$\lim_{z \to z_0} f(z) = w_0.$$

continuity:

If $f(z)$ is said to be continuous at $z = z_0$ then

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Derivative of a complex function:

A function $w = f(z)$ is said to be differentiable at a point z if the limit $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists.

It is denoted by. $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$.

Analytic or Holomorphic or Regular function:

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire function or Integral function:

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$. eg: e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$.

The necessary condition for f(x) to be Analytic:

Cauchy - Riemann Equations:

Statement:

The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

ie, $u_x = v_y$ and $v_x = -u_y$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function at the point z in a region R.

Since $f(z)$ is analytic its derivative $f'(z)$ exists in R and is given by,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \rightarrow (1).$$

Let $z = x + iy$

$\Delta z = \Delta x + i\Delta y$

ie, $z + \Delta z = (x+iy) + (\Delta x + i\Delta y)$

$= x + iy + \Delta x + i\Delta y$

$= (x + \Delta x) + i(y + \Delta y)$

Now, $f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$ $\rightarrow (2)$

Let us assume that $\Delta z \to 0$ by first assuming that $\Delta y = 0$ and then $\Delta x \to 0$.

ie, Δz is real. $[\Delta z = \Delta x + i\Delta y = \Delta x + i0 = \Delta x]$

Subs (2) in (1) we get,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - (u(x,y) + iv(x,y))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (3).$$

Now let us assume that $\Delta z \to 0$, by first assuming that $\Delta x = 0$ and then $\Delta y \to 0$ ie, Δz is imaginary.

$[\Delta z = \Delta x + i\Delta y = 0 + i\Delta y = i\Delta y, \therefore \Delta x = 0]$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{i} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \to (4)$$

Equating (3) & (4) we get, $-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

ie,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{or} \qquad u_x = v_y$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad\qquad u_y = -v_x.$$

The above equations are known as Cauchy-Riemann equations (or) C-R equations.

The sufficient condition for $f(z)$ to be analytic:

The function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if,

(i) $u(x,y)$ and $v(x,y)$ are differentiable in D and $u_x = v_y$ and $u_y = -v_x$

ii) The partial derivatives u_x, u_y, v_x and v_y are all continuous in D.

Polar form of C-R Equations:

Let $z = re^{i\theta}$ be the polar co-ordinates, then

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ ie, $u_r = \frac{1}{r}v_\theta$ $v_r = -\frac{1}{r}u_\theta$. where r is the modulus and θ is the argument

1. Test whether the function $f(z) = x^2 + iy^2$ is an analytic or not.

Sol: <u>Given:</u> $f(z) = x^2 + iy^2$

ie, $u + iv = x^2 + iy^2$

$$u = x^2 \qquad v = y^2$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 2y.$$

Here, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

Hence the given function is not an A.F.

2. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Sol: <u>Given:</u> $f(z) = \bar{z}$

Since, $z = x + iy$, $\bar{z} = x - iy$, $f(z) = u + iv$

$$\therefore u + iv = x - iy$$

ie, $u = x \qquad v = -y$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

Here, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

Hence, C-R equations are not satisfied anywhere.

Hence $f(z) = \bar{z}$ is nowhere differentiable.

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Q. Test the analyticity of the functions,

$f(z) = e^x(\cos y + i \sin y) = e^z$

Sol: Given: $f(z) = e^x(\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

$= u + iv$

where, $u = e^x \cos y \qquad v = e^x \sin y$

$u_x = \frac{\partial u}{\partial x} = e^x \cos y \qquad v_x = \frac{\partial v}{\partial x} = e^x \sin y$

$u_y = \frac{\partial u}{\partial y} = -e^x \sin y \qquad v_y = \frac{\partial v}{\partial y} = e^x \cos y.$

Here, $u_x = v_y$ and $u_y = -v_x$

ie, $f(z)$ satiesfies C-R equations.

The given function is analytic.

4. Show that $f(z) = \sin z$ is an analytic function.

Sol: Given: $f(z) = \sin z = \sin(x+iy)$ $[\because z = x+iy]$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$

$u + iv = \sin x \cosh y + i \cos x \sinh y$

ie, $u = \sin x \cosh y \qquad v = \cos x \sinh y$

$u_x = \frac{\partial u}{\partial x} = \cos x \cosh y \qquad v_x = \frac{\partial v}{\partial x} = -\sin x \sinh y$

$u_y = \frac{\partial u}{\partial y} = \sin x \sinh y \qquad v_y = \frac{\partial v}{\partial y} = \cos x \cosh y.$

$\therefore u_x = v_y, \quad u_y = -v_x.$

C-R equations are satisfied. Hence $f(z) = \sin z$ is an an.f.

5. Show that the function $f(z) = |z|^2$ is differentiable only at the origin.

Sol: Given: $f(z) = |z|^2 \rightarrow (1)$

Since, $z = x+iy$. we have $|z| = \sqrt{x^2+y^2}$

$$|z|^2 = x^2+y^2 \longrightarrow (2).$$

$|z|^2 = z\bar{z}$
$= (x+iy)(x-iy)$
$= x^2+y^2$
$|z| = \sqrt{x^2+y^2}$

Sub's (2) in (1) we get,

$$f(z) = x^2+y^2$$

ie, $u+iv = x^2+y^2$

ie, $u = x^2+y^2 \qquad v = 0.$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0.$$

If $f(z)$ is differentiable, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2y = 0 \Rightarrow y = 0.$$

$\therefore$ C-R Equations are satisfied only when $x=0, y=0$

Hence the given function $f(z)$ is differentiable only at the origin $(0,0)$.

6. Examine the analyticity of the function $f(z) = z^2$.

Sol: Given: $f(z) = z^2 = (x+iy)^2$

$$= x^2 - y^2 + 2ixy$$

$$= u + iv$$

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where $u = x^2 - y^2$ $\qquad$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\qquad$ $\frac{\partial v}{\partial y} = 2x$

Here, $u_x = v_y$ and $u_y = -v_x$.

The given function $f(z) = z^2$ is analytic.

7. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Sol: Given: $f(z) = (x + ay) + i(bx + cy)$

$= u + iv$

where $u = x + ay$ $\qquad$ $v = bx + cy$

$\frac{\partial u}{\partial x} = 1$ $\qquad$ $\frac{\partial v}{\partial x} = b$

$\frac{\partial u}{\partial y} = a$ $\qquad$ $\frac{\partial v}{\partial y} = c.$

Since $f(z) = u + iv$ is analytic, by C-R equations we get,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow c = 1$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow a = -b.$

8. If $u + iv$ is analytic show that $v - iu$ and $-v + iu$ are also analytic.

Sol: Given $u + iv$ is analytic.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Hence the C-R equations are satisfied.

$\therefore$ $f(z) = z^n$ is analytic.

10. Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Sol: Given: $f(z) = \cosh z = \cos(iz)$

$$= \cos[i(x+iy)]$$

$$= \cos(ix - y) = \cos ix \cos y + \sin(ix)\sin y$$

$$u + iv = \cosh x \cos y + i \sinh x \sin y.$$

$u = \cosh x \cos y \qquad v = \sinh x \sin y.$

$u_x = \sinh x \cos y \qquad v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y \qquad v_y = \sinh x \cos y$

$u_x = v_y$ and $v_x = -u_y$.

C-R equations are satisfied.

$\therefore$ $f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + i v_x$

$$= \sinh x \cos y + i \cosh x \sin y$$

$$= \sinh(x+iy)$$

$$= \sinh z.$$

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$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \rightarrow (1).$$

consider, $v - iu$

Here, $R.P = v$

$I.P = -u$

$$\left.\begin{aligned} v_x &= -u_y \Rightarrow u_y = -v_x \\ v_y &= -(-u_x) = u_x \Rightarrow u_x = v_y \end{aligned}\right\} \rightarrow (2).$$

From (2), $v - iu$ is analytic.

consider, $-v + iu$

Here, $R.P = -v$

$I.P = u$

$$\left.\begin{aligned} -v_x &= u_y \Rightarrow u_y = -v_x \\ -v_y &= -u_x \Rightarrow u_x = v_y \end{aligned}\right\} \rightarrow (3)$$

From (3), $v - iu$ is analytic.

Q. 9. Prove that $f(z) = z^n$ is analytic.

Sol: Let $z = re^{i\theta}$, $f(z) = z^n = r^n(e^{i\theta})^n$

$= r^n e^{in\theta}$

$= r^n(\cos n\theta + i\sin n\theta)$

$u = r^n \cos n\theta$ $\qquad$ $v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$ $\qquad$ $\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = -r^n \sin n\theta \cdot n$ $\qquad$ $\frac{\partial v}{\partial \theta} = r^n \cos n\theta \cdot n.$

Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$ is known as Laplace equation in two dimensions.

Laplacian operator:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ is called the Laplacian operator and is denoted by ∇^2.

Note: i) $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$ is known as Laplace equation in 3 dimensions.

ii) The Laplace equation in polar coordinates is defined as $\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0.$

Properties of Analytic functions

1. The real and imaginary parts of an analytic function $w = u + iv$ satisfy the Laplace equation in two dimensions viz $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

2. The real and imaginary parts of an analytic function $w = u(r,\theta) + iv(r,\theta)$ satisfy the Laplace equation in polar co-ordinates.

3. If $w = u(r,\theta) + iv(r,\theta)$ is an analytic function, the curves of the family $u(r,\theta) = a$ cut orthogonally the curves of the family $v(r,\theta) = b$, where a and b are arbitrary constants.

4. If $f(z)$ and $\overline{f(z)}$ are analytic in a region D. Show that $f(z)$ is constant in that region D.

5. If $w = u(x,y) + iv(x,y)$ is an analytic function the curves of the family $u(x,y) = a$ and the curves of the family $v(x,y) = b$ cut orthogonally where a and b are varying constants (or)

When the function $f(z) = u + iv$ is analytic, show that u = constant and V = constant are orthogonal.

Proof: Given $f(z)$ is an analytic function.

$\therefore$ By C.R equation.

$$u_x = v_y \text{ and } u_y = -v_x.$$

Given: $u(x,y) = a$ and $v(x,y) = b$.

consider the first curve $u(x,y) = a$; consider the 2nd curve $v(x,y) = b$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \qquad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \qquad \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\left(\frac{\partial v}{\partial y}\right)}$$

Then the slope of the 1st curve,

$$m_1 = \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{u_x}{u_y} \rightarrow (1)$$

Then the slope of the 2nd curve

$$m_2 = \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)} \quad [\text{by C-R}]$$

$$= m_2$$

Product of slopes at their point of intersection $= m_1 m_2$

$$= -\frac{\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)} = -1$$

[orthogonal $m_1 m_2 = -1$]

Hence the two families of curves form an orthogonal system.

6. An analytic function with constant modulus is constant.

Proof. Let $f(z) = u + iv$ be analytic.

By C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$|f(z)| = \sqrt{u^2+v^2} = c \neq 0$$

$$|f(z)|^2 = u^2+v^2 = c^2 \text{ (say)}$$

$$\text{i.e. } u^2+v^2 = c^2$$

diff. p.w.r. to x.

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \rightarrow (1)$$

diff. p.w.r to y

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0$$

$$-u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x} = 0$$

$$v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} = 0 \rightarrow (2) \text{ by C.R.}$$

$(1) \times u + (2) \times v \Rightarrow$

$$u^2 \frac{\partial u}{\partial x} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial u}{\partial x} - uv \frac{\partial v}{\partial x} = 0$$

$$(u^2+v^2) \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = 0 \quad [\because u^2+v^2 \neq 0]$$

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$$uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} - uv\frac{\partial u}{\partial x} + u^2\frac{\partial v}{\partial x} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial x} = 0 \qquad [\because u^2+v^2 \neq 0]$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0$$

$$f'(z) = 0$$

$\Rightarrow f(z) = c$ is constant

7. An analytic function whose real part is constant must itself be a constant. (or)

If $f(z)$ is analytic, show that $f(z)$ is constant if real part of $f(z)$ is constant.

Proof: Let $f(z) = u+iv$ be an analytic function

$\therefore$ By C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Given: u = constant

To Prove: $f(z)$ is a constant.

$$u = c$$

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

By CR equation

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0+i0$$

$$f'(z) = 0$$

$$f(z) = c$$

$\therefore f(z)$ is a constant.

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1. Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$

Sol: Let f be a function of x and y.

where x and y are function of z and $\bar{z}$.

Let $z = x + iy \rightarrow (1)$

$\bar{z} = x - iy \rightarrow (2)$

From (1) & (2) we get, $x = \frac{z+\bar{z}}{2}$

$y = \frac{z-\bar{z}}{2i} = -\frac{i}{2}(z-\bar{z})$.

$\frac{\partial x}{\partial z} = \frac{1}{2}$ $\quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$

$\frac{\partial y}{\partial z} = \frac{-i}{2}$ $\quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$

$$\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z} \rightarrow (A)$$

$$= \frac{\partial}{\partial x} \frac{1}{2} + \frac{\partial}{\partial y} \frac{-i}{2}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \rightarrow (3)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \rightarrow (B).$$

$$= \frac{\partial}{\partial x} \frac{1}{2} + \frac{\partial}{\partial y} \frac{i}{2}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \rightarrow (4)$$

(3) × (4) $\Rightarrow$ $$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \, \partial \bar{z}}$$

2. If $f(z)$ is a regular function of z, Prove that

(*) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2$$

Sol: Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2 + v^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(u^2) + \frac{\partial^2}{\partial y^2}(v^2) \rightarrow (1)$$

Now, $$\frac{\partial}{\partial x}(u^2) = 2u \frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \rightarrow (2)$$

Ily, $$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow (3).$$

(2) + (3) we get,

$$\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} = 2\left[u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

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$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\left(\left(\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial y^2}\right)\right)\right]$$

Since u is harmonic,

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad \left[\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right]$$

$$= 2\left[u_x^2 + u_y^2\right]$$

$$= 2\left[u_x^2 + (-v_x)^2\right] \quad \left[\because u_y = -v_x\right]$$

$$= 2\left[u_x^2 + v_x^2\right] \quad \left[\because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right.$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2 (u^2)}{\partial y^2} = 2|f'(z)|^2 \rightarrow (4)$$

Ily, $$\frac{\partial^2 (v^2)}{\partial x^2} + \frac{\partial^2 (v^2)}{\partial y^2} = 2|f'(z)|^2 \rightarrow (5)$$

Sub's (4) & (5) in (1) we get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2|f'(z)|^2 + 2|f'(z)|^2$$

$$= 4|f'(z)|^2.$$

Q. S.T $\nabla^2 \log|f(z)| = 0$. If $f(z)$ is analytic

Sol:

w.k.T $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

$$|f(z)|^2 = f(z)\overline{f(z)}$$

$$|f(z)| = \sqrt{f(z)\overline{f(z)}}$$

$$\log|f(z)| = \log\sqrt{f(z)\overline{f(z)}}$$

$$= \frac{1}{2}\left[\log f(z) + \log \overline{f(z)}\right]$$

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$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\ \frac{1}{2}\left[\log f(z) + \log \overline{f(z)}\right]$$

$$= 2\ \frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\log f(z) + \log \overline{f(z)}\right]$$

$$= 2\ \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left[\log f(z) + \log \overline{f(z)}\right]$$

$$= 2\ \frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\log f(z)\right] + 2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\log \overline{f(z)}\right]$$

$$= 2\ \frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\log f(z)\right] + 2\frac{\partial}{\partial \bar{z}}\left[\overline{\frac{\partial}{\partial z}\log f(z)}\right]$$

$$= 2\ \frac{\partial}{\partial z}[0] + 2\frac{\partial}{\partial \bar{z}}[0]$$

$$= 0$$

Since $\log f(z)$ is analytic.

$\therefore$ $\log f(z)$ is independent of $\bar{z}$ and

$\overline{\frac{\partial}{\partial z}\log f(z)}$ is independent of z.

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HARMONIC Function:

Any function which has continuous second order partial derivatives and which satisfies Laplace's equation is called Harmonic function.

1. S.T $u = 2x(1-y)$ is harmonic.

Sol: Given $u = 2x(1-y)$

W.k.T the Laplace equation $\nabla^2 u = 0$.

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u = 2x(1-y)$

$\dfrac{\partial u}{\partial x} = 2(1-y)$ $\quad\quad$ $\dfrac{\partial u}{\partial y} = -2x$

$\dfrac{\partial^2 u}{\partial x^2} = 0$ $\quad\quad$ $\dfrac{\partial^2 u}{\partial y^2} = 0$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic.

2. P.T $u = e^x \cos y$ is a harmonic function.

Sol: Given: $u = e^x \cos y$

$\dfrac{\partial u}{\partial x} = e^x \cos y$ $\quad\quad$ $\dfrac{\partial u}{\partial y} = -e^x \sin y$

$\dfrac{\partial^2 u}{\partial x^2} = e^x \cos y$ $\quad\quad$ $\dfrac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$

$\therefore$ u is harmonic.

Conjugate harmonic function:

If u and v are harmonic functions such that $u+iv$ is an analytic function are called the conjugate harmonic functions.

Construction of Analytic functions:

1. Exact differential method:

i) Suppose the harmonic function $u(x,y)$ is given.

Now $dv = v_x\, dx + v_y dy$ is an exact differential where v_x and v_y are known from u by using C.R equations

$$v = \int v_x\, dx + \int v_y\, dy = -\int u_y\, dx + \int u_x\, dy.$$

ii) Suppose the harmonic function $v(x,y)$ is given.

Now $du = u_x\, dx + u_y\, dy$ is an exact differential where u_x and u_y are known from v by using C.R equations

$$u = \int u_x\, dx + \int u_y\, dy = \int v_y\, dx + \int -v_x\, dy$$

$$= \int v_y\, dx - \int v_x\, dy.$$

2. Milne Thomson Method:

To find $f(z)$ when u is given,

W.K.T $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad [\text{by C.R}] \rightarrow (1)$$

Assume that. $\frac{\partial u}{\partial x}(x,y) = \varphi_1(z,0)$, $\frac{\partial u}{\partial y}(x,y) = \varphi_2(z,0)$. $\rightarrow (2)$

Sub's ② in ① we get,

$$f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$$

$$\int f'(z)\,dz = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$$

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz + C.$$

where C is a complex constant.

To find $f(z)$ when v is given:

W.K.T $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{by C.R} \longrightarrow ①$$

Assume that,

$$\left.\begin{aligned} \frac{\partial v(x,y)}{\partial y} &= \varphi_1(z,0) \\ \frac{\partial v(x,y)}{\partial x} &= \varphi_2(z,0) \end{aligned}\right\} \rightarrow ②$$

Sub ② in ① we get,

$$f'(z) = \varphi_1(z,0) + i\varphi_2(z,0)$$

$$\int f'(z)\,dz = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz$$

$$f(z) = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz + C$$

where C is a complex constant.

To find $f(z)$ when u is given:

1. Determine the analytic function where real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Sol: Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\varphi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\varphi_1(z,0) = 3z^2 + 6z$$

$$\varphi_2(x,y) = \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\varphi_2(z,0) = 0.$$

By Milne's Thomson method.

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz.$$

$$= \int (3z^2 + 6z)\,dz - 0$$

$$= \frac{3z^3}{3} + \frac{\overset{3}{\cancel{6}}\,z^2}{2} + c$$

$$f(z) = z^3 + 3z^2 + c.$$

2. If $u = e^x \sin y$, find $f(z) = u + iv$.

Sol: Given: $u = e^x \sin y$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \sin y$$

$$\therefore \phi_1(z,0) = e^z \sin 0 = 0$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^x \cos y$$

$$\therefore \phi_2(z,0) = e^z \cos 0 = e^z$$

By Milne's Thomson method.

$$f(z) = \int \phi_1(z,0)dz - i\int \phi_2(z,0)dz$$

$$= \int 0\, dz - i\int e^z\, dz$$

$$= -\int i e^z\, dz$$

$$= -ie^z + C.$$

3. Find $f(z) = u+iv$, if $u = e^x \cos y$

Sol: Given: $u = e^x \cos y$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\therefore \phi_1(z,0) = e^z \cos 0 = e^z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\therefore \phi_2(z,0) = 0$$

By Milne's Thomson. $f(z) = \int \phi_1(z,0)dz - i\int \phi_2(z,0)dz$

$$= \int e^z\, dz = e^z + C.$$

4. find $f(z)$, if $u(x,y) = e^x(x\cos y - y\sin y)$.

Sol: Given: $u = e^x(x\cos y - y\sin y) = e^x . x\cos y - e^x y\sin y$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \cos y\,(xe^x + e^x) - y\sin y . e^x$$

$$\phi_1(z,0) = \cos 0\,(ze^z + e^z) - 0 . \sin 0\, e^z$$

$$= ze^z + e^z$$

$\phi_2(x,y) = \frac{\partial u}{\partial y} = -xe^x \sin y - e^x(\sin y + y\cos y)$

$\phi_2(z,0) = 0$

By Milne's Thomson method.

$$f(z) = \int \phi_1(z,0)\,dz + \int \phi_2(z,0)\,dz$$

$$= \int (z e^z + e^z)\,dz + 0$$

$$= \int e^z (z+1)\,dz$$

$$= (z+1)e^z - (1)e^z + C$$

$$= ze^z + e^z - e^z + C$$

$$= ze^z + C.$$

5. Show that the function $u = 2xy + 3y$ is harmonic and find the corresponding analytic function. Find its conjugate.

Sol: Given $u = 2xy + 3y$

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2x + 3$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfies Laplace's equation

$\therefore$ u is harmonic.

By Milne's Thomson method, we have, (13)

Given: $u = 2xy + 3y$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = 2y$$

$$\therefore \phi_1(z,0) = 2.0 = 0.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = 2x + 3$$

$$\therefore \phi_2(z,0) = 2z + 3$$

By M.T method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int 0\,dz - i\int (2z+3)\,dz$$

$$= -i\left(\frac{\cancel{2}z^2}{\cancel{2}} + 3z\right) + C$$

$$f(z) = -i(z^2 + 3z) + C$$

where C is a complex constant.

To find conjugate of u.

w.k.T $\quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$\int dv = \int \frac{\partial v}{\partial x}dx + \int \frac{\partial v}{\partial y}dy$$

$$\int dv = \int -\frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dx.$$

$$\int dv = \int -(2x+3)dx + \int 2y\,dy.$$

$$v = -\left(\cancel{2}x^2/\cancel{2} + 3x\right) + \cancel{2}y^2/\cancel{2}$$

$$v = -x^2 - 3x + y^2 = y^2 - 3x - x^2.$$

6. Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find $f(z)$.

(4)

Sol: <u>Given :</u> $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) \qquad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$$

$$= \frac{x}{x^2+y^2} \qquad = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \qquad = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \qquad = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfies the Laplace equation.

$\therefore$ u is harmonic.

<u>To find Conjugate of u :</u>

i.e, <u>To find v</u>

W.k.T $\quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$ [Exact D.M]

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$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \quad (\text{by C-R})$$

$$= -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = \frac{x\,dy - y\,dx}{x^2\left(1+y^2/x^2\right)}$$

$$= \frac{x\,dy - y\,dx}{x^2}\cdot\frac{1}{\left(1+(y/x)^2\right)}$$

$$dv = \frac{1}{1+(y/x)^2}\,d(y/x)$$

$$\int dv = \int \frac{d(y/x)}{1+(y/x)^2}$$

$$v = \int u_y\,dx + \int u_x\,dy$$

$$= -\int \frac{y}{x^2+y^2}\,dx + \int \frac{x}{x^2+y^2}\,dy$$

$$\boxed{v = \tan^{-1}(y/x)}$$

<u>To find $f(z)$:</u>

$$f(z) = u + iv$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$= \log(x+iy)$$

$$= \log z$$

$[\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta$

$z = x+iy$ $\qquad \tan\theta = y/x$ $\qquad \log r = \frac{1}{2}\log(x^2+y^2)$

$r = |z| = \sqrt{x^2+y^2}$ $\qquad \theta = \tan^{-1}(y/x)$

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7. Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol: Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(\cosh 0 - \cos 2z)(2\cos 2z) - 2\sin^2 2z}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{\left[2\cos 2z - 2(1+\cos 2z)\right]\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{2\cancel{\cos 2z} - 2 - 2\cancel{\cos 2z}}{1-\cos 2z} = \frac{-2}{1-\cos 2z}$$

$$= -\frac{1}{\left(\frac{1-\cos 2z}{2}\right)} = -\frac{1}{\sin^2 z} = -\operatorname{cosec}^2 z.$$

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$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x[2\sinh 2y]}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \frac{-2\sin 2z \sinh 20}{(\cosh 20 - \cos 2z)^2}$$ $\cosh 0 = 1$, $\sinh 0 = 0$

$$\phi_2(z,0) = 0$$

By M-T Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int -\operatorname{cosec}^2 z\,dz - 0$$

$$f(z) = \cot z + c$$

To find $f(z)$ when v is given:

8. Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Sol: $$v = e^{-x}(x\cos y + y\sin y)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y.$$

$$\phi_1(x,y) = \frac{\partial v}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y(1))$$

$$\phi_1(z,0) = e^{-z}[0+0+0] = 0.$$

$$\phi_2(x,y) = \frac{\partial v}{\partial x} = \cos y\left(x - e^{-x} + e^{-x}(1)\right) + y\sin y \cdot e^{-x}$$

$$= -e^{-x}x\cos y + e^{-x} - e^{-x}y\sin y.$$

$$\phi_2(z,0) = -e^{-z}z\cos 0 + e^{-z} - e^{-z}\,0\sin 0$$

$$= -e^{-z}z + e^{-z}$$

$$= e^{-z}(1-z)$$

By M.T method,

$$f(z) = \int \phi_1(z,0)\,dz + i\,\phi_2(z,0)\,dz.$$

$$= \int 0\,dz + i\int(1-z)e^{-z}dz$$

$\int u\,dv = uv - \int v\,du$. $u = (1-z)$, $dv = e^{-z}dz$; $du = -dz$, $v = \frac{e^{-z}}{-1}$

$$= i\left[(1-z)\left(\frac{e^{-z}}{-1}\right) - \int\left(\frac{e^{-z}}{-1}\right) - dz\right]$$

$$= i\left[-(1-z)e^{-z} - \left(\frac{e^{-z}}{-1}\right) + c\right]$$

$$= i\left[\cancel{-e^{-z}} + ze^{-z} + \cancel{e^{-z}} + c\right]$$

$$= i\left[ze^{-z}\right] + c.$$

Q. Find an analytic function whose imaginary part is $3x^2y - y^3$.

Sol: Given: I.M $v = 3x^2y - y^3$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\phi_1(x,y) = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$\therefore \phi_1(z,0) = 3z^2 - 3(0)^2 = 3z^2$

$\phi_2(x,y) = \frac{\partial v}{\partial x} = 6xy$

$\therefore \phi_2(z,0) = 6z.0 = 0$

By M-T Method,

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$= \int 3z^2\,dz + i\int 0\,dz$$

$$= \cancel{3}\,z^3/\cancel{3} + C$$

$$f(z) = z^3 + C$$

To find f(z) when u+v or u-v is given:

10. If $f(z) = u+iv$ is an analytic function and $u-v = e^x(\cos y - \sin y)$ find $f(z)$ interms of z.

Sol: Given : $f(z) = u+iv \rightarrow (1)$

i.e, $u+iv = f(z)$

$if(z) = iu - v \longrightarrow (2)$

$(1) + (2) \Rightarrow (1+i)f(z) = u+iv+iu-v$

$$= (u-v) + i(u+v)$$

$$F(z) = U + iV$$

where, $F(z) = (1+i)f(z)$.

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$$U = u - v$$
$$V = u + v$$
$$U = u - v = e^x(\cos y - \sin y)$$
$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$
$$\phi_1(z,0) = e^z(\cos 0 - \sin 0) = e^z(1-0)$$
$$= e^z$$
$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x[-\sin y - \cos y]$$
$$\phi_2(z,0) = e^z[-\sin 0 - \cos 0] = e^z[0-1] = -e^z$$
$$= -e^z$$

By Milne's Thomson method,

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$
$$= \int e^z\,dz - i\int -e^z\,dz$$
$$F(z) = e^z + i\,e^z + C_1$$
$$(1+i)\,f(z) = e^z(1+i) + C_1$$
$$f(z) = e^z + C.$$

11. Find the analytic function $f(z) = u+iv$ given that $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ (17)

Sol: Given: $f(z) = u+iv \rightarrow (1)$

$i f(z) = iu - v \rightarrow (2)$

Adding (1) & (2) we get,

$$f(z) + i f(z) = u + iv + iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

where, $F(z) = (1+i) f(z)$

$$U = u - v, \quad V = u+v.$$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

Now, $V = u+v = \dfrac{\sin 2x}{(\cosh 2y - \cos 2x)}$

$$\phi_1(x,y) = \frac{\partial V}{\partial y} = \frac{(\cosh 2y - \cos 2x)\cdot 0 - \sin 2x\,[2\sinh 2y - 0]}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{-2\sin 2z \sinh 0}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{0}{(1-\cos 2x)^2} = 0 \quad [\because \sinh 0 = 0]$$

$$\phi_1(z,0) = 0$$

$$\phi_2(x,y) = \frac{\partial v}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,[0-(-2\sin 2x)]}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \frac{(\cosh 0 - \cos 2z)(2\cos 2z) - \sin 2z(2\sin 2z)}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2[\cos 2z - (\cos^2 2z + \sin^2 2z)]}{(1-\cos 2z)^2}$$

$$= \frac{2[\cos 2z - 1]}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} \quad [1-\cos 2\theta = 2\sin^2\theta]$$

$$= \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(z,0) = -\operatorname{cosec}^2 z.$$

$$\therefore \quad F(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$F(z) = 0 + i\int(-\operatorname{cosec}^2 z)\,dz = i\cot z + C_1$$

Where, $C = \frac{C_1}{1+i}$

$$(1+i)f(z) = i\cot z + C_1$$

$$f(z) = \frac{i}{1+i}\cot z + \frac{C_1}{1+i} = \frac{i}{(1+i)} \times \frac{(1-i)}{(1-i)}\cot z + C = \frac{i-i^2}{1+1}\cot z + C$$

$$\boxed{\therefore f(z) = \frac{1+i}{2}\cot z + C}$$

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Complex potential functions:

In two dimensional steady state flows. the Complex potential function as.

$$f(z) = \phi(x,y) + i\,\psi(x,y).$$

Here, the real part $\phi(x,y)$ is called Velocity potential function or velocity potential and $\psi(x,y)$ is called the stream function or lines of force.

1. In a two dimensional fluid flow, the Stream function is $\chi = \tan^{-1}(y/x)$. Find the velocity potential ϕ.

Sol: Let $f = \phi + i\psi$

Given : $\chi = \tan^{-1}(y/x)$

$$\phi_1(x,y) = \frac{\partial \chi}{\partial y} = \left(\frac{1}{1+(y^2/x^2)}\right)\left(\frac{x.1 - y.0}{x^2}\right)$$

$$= \frac{x}{\cancel{x^2}\left(\frac{x^2+y^2}{\cancel{x^2}}\right)} = \frac{x}{x^2+y^2}$$

$$\Rightarrow \phi_1(z,0) = \frac{\cancel{z}}{z^{\cancel{2}}+0} = \frac{1}{z}.$$

$$\phi_2(x,y) = \frac{\partial \chi}{\partial x} = \frac{1}{(1+y^2/x^2)}\left(\frac{x.0 - y.1}{x^2}\right) = \frac{-y}{\cancel{x^2}\left(\frac{x^2+y^2}{\cancel{x^2}}\right)}$$

$$\Rightarrow \phi_2(z,0) = \frac{-0}{z^2+0^2} = 0.$$

$$\therefore f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

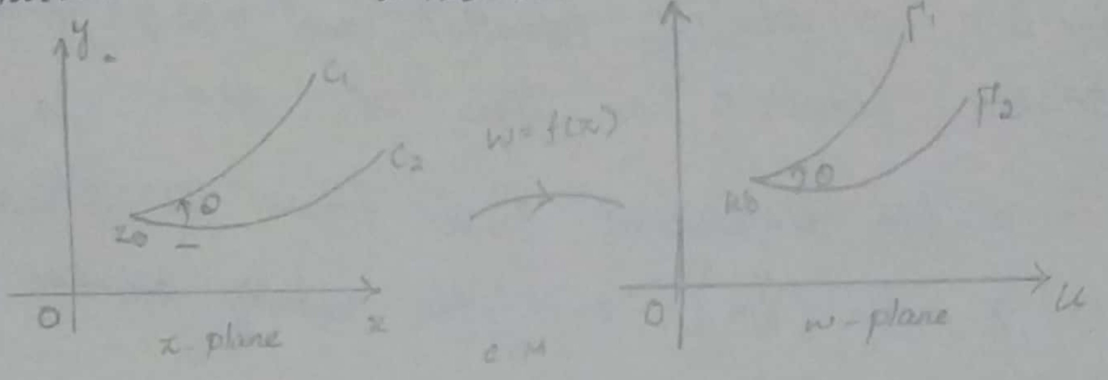
$$= \int \frac{1}{z}\,dz = \log z = \log(x+iy)$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$\therefore$ velocity potential $= \frac{1}{2}\log(x^2+y^2)$.

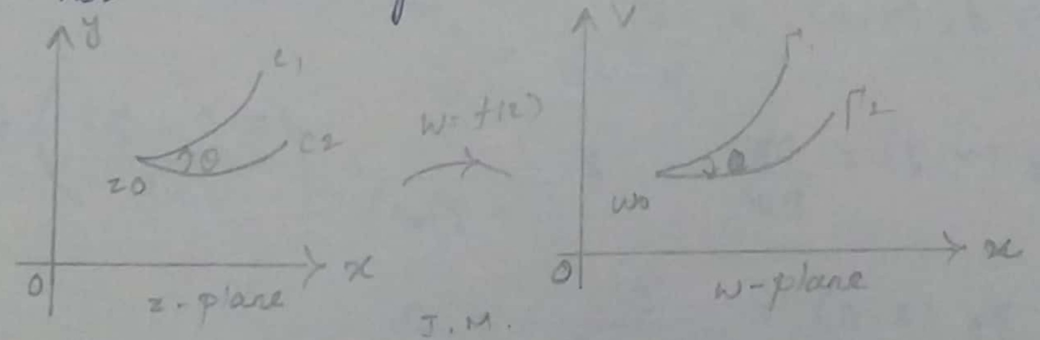
Conformal Mapping:

A Mapping $w = f(z)$ is said to be conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.



Isogonal Mapping:

A Mapping $w = f(z)$ is said to be isogonal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane only in magnitude but not necessarily in direction.



NOTE:

(i) A Mapping $w = f(z)$ is said to be conformal at $z = z_0$, if $f'(z_0) \neq 0$.

(ii) The point, at which the mapping $w = f(z)$ is not conformal, i.e., $f'(z) = 0$ is called a critical point of the mapping.

The Critical points of $z = f'(w)$ are given by $\frac{dz}{dw} = 0$.

Hence the critical points of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

Fixed points of Mapping:

Fixed or invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves, are "kept fixed" under the mapping.

Thus they are obtained from $w = f(z) = z$.

Identity mapping:

The identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points.

1. Find the critical points of the transformation $w = z + \frac{1}{z}$.

$d(x^n) = n x^{n-1}$

$\frac{1}{z} = z^{-1} = -1 z^{-1-1}$
$= -1 z^{-2}$
$= -\frac{1}{z^2}$

Sol: Let $w = z + \frac{1}{z}$

$$\Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2}.$$

$$\therefore \frac{dz}{dw} = \frac{z^2}{z^2 - 1}$$

The critical points occur at $\frac{dw}{dz} = 0 \Rightarrow \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^2 - 1}{z^2} = 0, \quad \frac{z^2}{z^2 - 1} = 0$$

$\Rightarrow z^2 - 1 = 0 \quad z^2 = 0$ $\therefore$ The c.p are $z = \pm 1, 0$.

$\Rightarrow z = \pm 1 \quad z = 0$

The Transformation $w = c + z$.

1. Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$

$w = z + c$

Sol: Let $z = x + iy$

$w = u + iv$

Given: $w = z + 3 + 2i$

$\Rightarrow u + iv = z + 3 + 2i.$

$\Rightarrow u + iv = (x + iy) + (3 + 2i)$

$u + iv = (x + 3) + i(y + 2)$

$u = x + 3, \quad v = y + 2.$

Given the circle $|z| = 2$

$|z|^2 = 4$

$x^2 + y^2 = 4$

$\left[\begin{array}{ll} u = x + 3 & v = y + 2 \\ \therefore x = u - 3 & y = v - 2 \end{array} \right]$

$z = x + iy$

$|z|^2 = x^2 + y^2$

ie, $(u-3)^2 + (v-2)^2 = 4$

$(u-a)^2 + (v-b)^2 = r^2 \quad r = 2$

(a, b)

Hence the circle $x^2 + y^2 = 4$ mapped into $(u-3)^2 + (v-2)^2 = 4$ in w-plane which is also a circle, with centre at $(3, 2)$ and radius 2.

2. Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$ [Ans: $c = (3, 2)$, $r = 1$]

$|z|^2 = 1$

$x^2 + y^2 = 1$

The Transformation $w = cz$

1. Find the image of the circle $|z| = r$ under the transformation $w = 5z$.

Sol.

Given: $W = 5z$

$$u+iv = 5(x+iy) = 5x + i5y.$$

$$u = 5x \quad v = 5y$$

$$\Rightarrow x = \frac{u}{5} \quad y = \frac{v}{5} \rightarrow (1)$$

Given: $|z| = r$ $\quad |z|^2 = r^2$

$$\Rightarrow x^2+y^2 = r^2 \rightarrow (2)$$

Sub's (1) in (2) weget.

$$\left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = r^2$$

$$u^2+v^2 = 25r^2 = (5r)^2$$

$$(u-0)^2+(v-0)^2 = 25r^2$$

which is a circle in w-plane whose centre is at (0,0) and radius is $5r$.

Hence $|z|=r$ is tranformed into $|w| = 5r$.

The transformation $w = \frac{1}{z}$

Let $z = x+iy$
$w = u+iv$

Given: $w = \frac{1}{z}$ or $z = \frac{1}{w}$

ie., $$x+iy = \frac{1}{u+iv} = \frac{1}{(u+iv)} \times \frac{(u-iv)}{(u-iv)}$$

$(a+b)(a-b) = a^2-b^2$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$u^2-(iv)^2$
u^2+v^2

Equating the real and imaginary parts, weget,

$$x = \frac{u}{u^2+v^2} \rightarrow (1) \quad y = \frac{-v}{u^2+v^2} \rightarrow (2).$$

W.K.T The general equation of circle in z-plane is,

$$x^2+y^2+2gx+2fy+c = 0. \rightarrow (3)$$

Sub's ① & ② in ③ we get.

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\frac{u}{u^2+v^2} + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0.$$

$$\Rightarrow c(u^2+v^2) + 2gu - 2fv + 1 = 0. \rightarrow ④$$

which is the equation of the circle in w-plane. Hence under the transformation $w = \frac{1}{z}$ a circle in z-plane transformes to another circle in w-plane. when the circle passes through the origin we have $c=0$ in ③. when $c=0$, equation ④ gives a straight line.

1. Find the image of $x=2$ under the transformation $w=\frac{1}{z}$.

Sol: Given $w = \frac{1}{z}$

ie., $z = \frac{1}{w}$

$$x+iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

ie, $x+iy = \dfrac{u-iv}{u^2+v^2}$

$$x = \frac{u}{u^2+v^2} \rightarrow ① \qquad y = \frac{-v}{u^2+v^2} \rightarrow ②$$

when $x=2$,

$$① \Rightarrow 2 = \frac{u}{u^2+v^2}$$

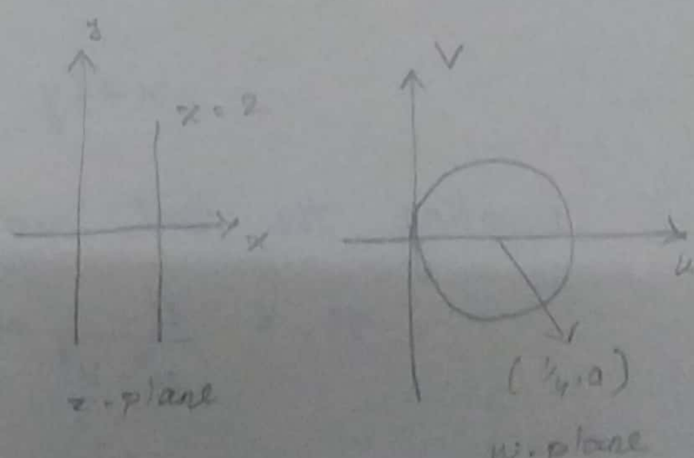
$$2(u^2+v^2) = u$$

$$u^2+v^2 = \frac{u}{2}$$

$$\Rightarrow u^2+v^2-\frac{u}{2} = 0$$

$$\left(u-\frac{1}{4}\right)^2+v^2-\frac{1}{16} = 0$$

$$\Rightarrow \left(u-\frac{1}{4}\right)^2+(v-0)^2 = \frac{1}{16} = \left(\frac{1}{4}\right)^2$$



which is a circle whose centre is $(1/4, 0)$ and radius is $\frac{1}{4}$. (21)

$\therefore$ The line $x = 2$ in the z-plane is transformed into a circle.

$\therefore \left(u - \frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$ in the w-plane.

2. Find the image of $|z - 2i| = 2$, under the transformation $w = 1/z$.

Sol: Given the transformation is $w = 1/z$.

ie, $z = \frac{1}{w}$

Given curve is $|z - 2i| = 2$

$$|x + iy - 2i| = 2$$

$$\Rightarrow |x + i(y-2)| = 2$$

$$\Rightarrow x^2 + (y-2)^2 = 2^2$$

$$\Rightarrow x^2 + y^2 + 4 - 4y = 4$$

$$x^2 + y^2 - 4y = 0 \rightarrow ①.$$

Now $w = u + iv$

$$\therefore z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u - iv}{u^2+v^2}$$

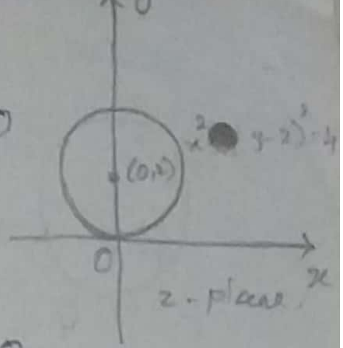
$$x = \frac{u}{u^2+v^2} \rightarrow ② \qquad y = \frac{-v}{u^2+v^2} \rightarrow ③$$

Subs ② & ③ in ① we get,

$$x^2 + y^2 - 4y = 0$$

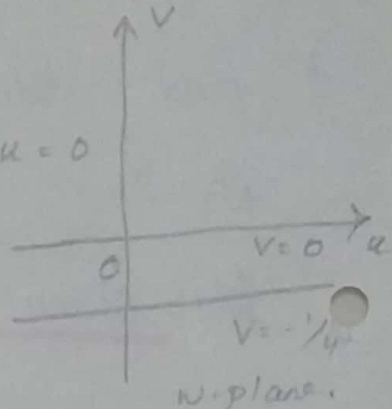
$$\left(\frac{u}{u^2+v^2}\right)$$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$



$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{(u^2+v^2)} = 0$$

$$\cancel{\frac{u^2(u^2+v^2)+v^2(u^2+v^2)+4v}{(u^2+v^2)}} = 0$$



$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$(u^2+v^2)+4v(u^2+v^2) = 0$$

$$(u^2+v^2)(1+4v) = 0 \qquad [\because u^2+v^2 \neq 0]$$

$$1+4v = 0$$

which is a straight line in w-plane.

3. Find the image of the infinite strips

i) $\frac{1}{4} < y < \frac{1}{2}$ ii) $0 < y < \frac{1}{2}$ under the transformation

$w = \frac{1}{z}$.

Sol: Given: $w = \frac{1}{z}$

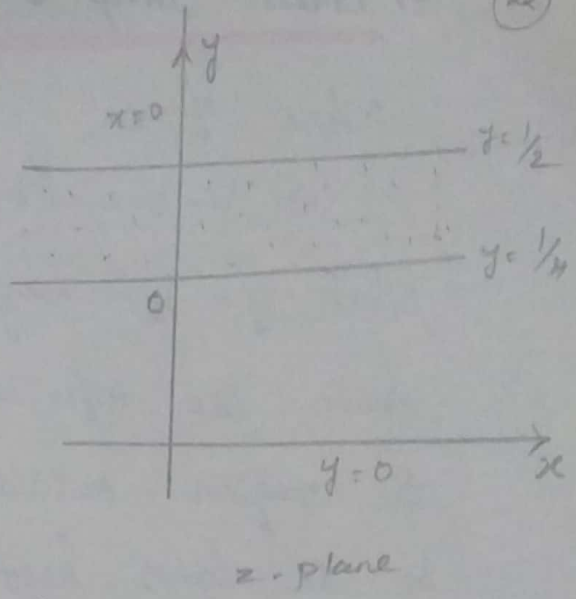
ie, $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

ie, $x = \frac{u}{u^2+v^2} \rightarrow (1)$ $\quad y = \frac{-v}{u^2+v^2} \rightarrow (2)$

i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$:



when $y = \frac{1}{4}$

From ②, $\frac{1}{4} = \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2 = -4v$

$\rightarrow u^2+v^2+4v = 0$

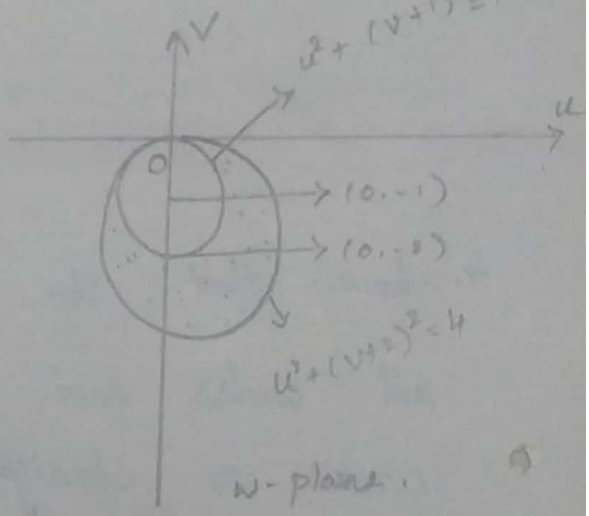
$\Rightarrow u^2+(v+2)^2-4 = 0$

$u^2+(v+2)^2 = 4 \rightarrow$ ③

$(u-a)^2+(v-b)^2 = r^2 \qquad (u-0)^2+(v-(-2))^2 = 2^2$

which is a circle whose centre is at $(0,-2)$ in the w plane and radius is 2.

when $y = \frac{1}{2}$



From ②, $\frac{1}{2} = \frac{-v}{u^2+v^2}$

$u^2+v^2 = -2v$

$u^2+v^2+2v = 0$

$u^2+(v+1)^2-1 = 0$

$u^2+(v+1)^2 = 1 \rightarrow$ ④

which is a circle whose centre is at $(0,-1)$ in the w plane and radius is 1.

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region inbetween circles.

$u^2+(v+1)^2 = 1$ and $u^2+(v+2)^2 = 4$ in the w-plane.

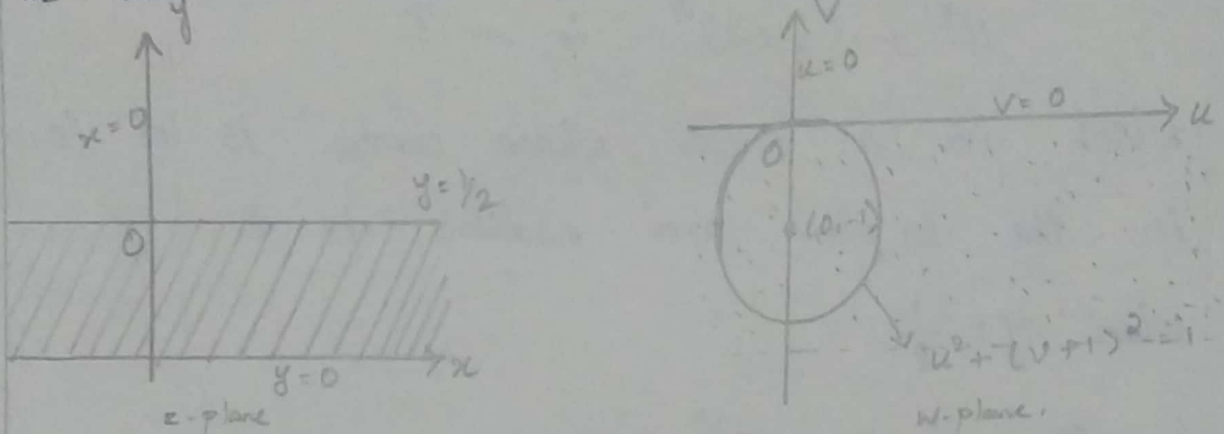
ii) Given strip is $0 < y < \frac{1}{2}$:

When $y = 0$

$\Rightarrow v = 0$ [by ②]

When $y = \frac{1}{2}$ we get, $u^2 + (v+1)^2 = 1$ by ④

Hence the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w-plane.



4. Show that the transformation $w = \frac{1}{z}$ transforms all circles and straight lines in the z-plane into circles or straight lines in the w-plane.

Sol: Given: $w = \frac{1}{z}$

ie., $z = \frac{1}{w}$

Now, $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{1}{u+iv}\,\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

ie, $$x + iy = \frac{u}{u^2+v^2} - i\,\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \rightarrow ① \quad y = \frac{-v}{u^2+v^2} \rightarrow ②$$

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The general equation is,

$$a(x^2+y^2)+2gx+2fy+c=0 \rightarrow ③$$

$$a\left[\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right]+2g\left[\frac{u}{u^2+v^2}\right]+2f\left[\frac{-v}{u^2+v^2}\right]+c=0.$$

$$\frac{a(u^2+v^2)}{(u^2+v^2)^2}+2g\frac{u}{u^2+v^2}-2f\frac{v}{u^2+v^2}+c=0.$$

The transformed equation is,

$$c(u^2+v^2)+2gu-2fv+a=0 \longrightarrow ④$$

i) $a\neq 0$, $c\neq 0 \Rightarrow$ circles not passing through the origin in z-plane map into circles not passing through the origin.

ii) $a\neq 0$, $c=0 \Rightarrow$ circles through the origin in z-plane map onto straight lines not through the origin.

iii) $a=0$, $c\neq 0 \rightarrow$ the straight lines not through the origin in z-plane map onto circles through the origin in w-plane.

(iv) $a=0$, $c=0 \Rightarrow$ straight lines through the origin of z-plane onto straight lines through the origin in w-plane.

BILINEAR TRANSFORMATION: [or Mobius Transformation].

The transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

Fixed points or invariant points:

The fixed points of the bilinear transformation $w = \dfrac{az+b}{cz+d}$ is given by $z = \dfrac{az+b}{cz+d}$.

Cross ratio:

The cross-ratio of the four points z_1, z_2, z_3, z_4 in the z-plane is defined by $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

NOTE:

The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$.

1. Find the fixed points of $w = \dfrac{2zi+5}{z-4i}$

Sol: The fixed points are given by. $[w = z]$

$$z = \frac{2zi+5}{z-4i}$$

$\Rightarrow z(z-4i) = 2zi+5$

$\Rightarrow z^2 - 4zi - 2zi - 5 = 0$

$\Rightarrow z^2 - 6zi - 5 = 0.$

$i^2 = -1$

$a = 1$
$b = -6i$
$c = -5$

$(-6i)^2 = 36i^2 = 36(-1) = -36$

$\sqrt{-16} = \sqrt{i^2 16} = i4$

$$z = \frac{6i \pm \sqrt{-36+20}}{2}$$

$$z = \frac{6i \pm \sqrt{-16}}{2} = \frac{6i \pm 4i}{2}$$

$$z = \frac{6i+4i}{2} \qquad z = \frac{6i-4i}{2}$$

$$= \frac{\overset{5}{\cancel{10}}i}{\cancel{2}} = 5i \qquad z = \frac{\cancel{2}i}{\cancel{2}} = i$$

$$\therefore \quad z = 5i, i.$$

2. Find the invariant points of the B.T $\frac{z-1}{z+1}$

Sol: The invariant points are given by,

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z-1$$

$$z^2 + z = z-1$$

$$\Rightarrow z^2 + \cancel{z} - \cancel{z} + 1 = 0 \quad \Rightarrow z^2 + 1 = 0 \Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

$$\therefore \quad z = i, -i.$$

3. Determine the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in the w plane.

Sol: Given: $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

$w_1 = 0$, $w_2 = i$, $w_3 = 3i$

Let the B.T be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

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$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)\,i} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{-2w}{(w-3i)} = -\frac{z+1}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow \quad 2w(z-1) = (w-3i)(z+1)$$

$$\Rightarrow \quad 2wz - 2w = wz + w - 3zi - 3i$$

$$2wz - 2w - wz - w = -3i(z+1)$$

$$w[2z-2-z-1] = -3i(z+1)$$

$$w[z-3] = -3i(z+1)$$

$$w = \frac{-3i(z+1)}{(z-3)}.$$

2. Find the Bilinear transformation that maps the points $1+i, -i, 2-i$ of the z-plane into the points $0, 1, i$ of the w-plane.

Sol: Given: $z_1 = 1+i \quad z_2 = -i \quad z_3 = 2-i$

$w_1 = 0 \quad w_2 = 1 \quad w_3 = i$

Let the B.T be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$w = \frac{2i(z-1-i)}{2z-2-2i+6-3i-3z+2i+1-iz}$$

$$w = \frac{2i(z-1-i)}{-z+5-3i-iz}$$

3. Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Sol: Given: $z_1 = \infty \quad z_2 = i \quad z_3 = 0$

$w_1 = 0 \quad w_2 = i \quad w_3 = \infty$

Let the B.T be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\,\cancel{z_1}\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{\left(\frac{w}{\infty}-1\right)(i-0)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{(z-0)\left(\frac{i}{\infty}-1\right)}$$

$$\frac{w(0-1)}{(0-1)i} = \frac{(0-1)(i)}{(z-0)(0-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z} \Rightarrow \frac{w}{i} = \frac{i}{z}$$

$\Rightarrow \quad wz = i^2 = -1$

$\Rightarrow \quad w = \frac{-1}{z}. \quad \Rightarrow z = -1/z \Rightarrow z^2 = -1$

$z = \pm i$

(55)

$$\frac{(w-0)(1-i)}{(0-1)\cancel{(i-w)}} = \frac{\cancel{[z-(1+i)][-i-(2-i)]}}{[(1+i)+i][(2-i)-z]}$$

$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{[z-(1+i)][-i-(2-i)]}{[z-(2-i)][-i-(1+i)]}$$

$$\frac{w(1-i)}{(w-i)\cdot 1} = \frac{[z-1-i][-\cancel{i}-2+\cancel{i}]}{[z-2+i][-i-1-i]}$$

$$\frac{w(1-i)}{(w-i)} = \frac{-2(z-1-i)}{(z+i-2)(-1-2i)}$$

$$\frac{w}{w-i} = \frac{-2}{(1-i)}\frac{(z-1-i)}{(1+2i)(2-i-z)}$$

$$= \frac{-2(z-1-i)}{(1+2i-i+2)(2-i-z)} = \frac{-2(z-1-i)}{(3+i)(2-i-z)}$$

$$\therefore \frac{w-i}{w} = -\frac{(3+i)(2-i-z)}{2(z-1-i)}$$

$$1-\frac{i}{w} = -\frac{(3+i)(2-i-z)}{2(z-1-i)}$$

$$-\frac{i}{w} = -\frac{(3+i)(2-i-z)}{2(z-1-i)} - 1$$

$$\frac{i}{w} = 1 + \frac{(3+i)(2-i-z)}{2(z-1-i)}$$

$$\frac{i}{w} = \frac{2(z-1-i)+(3+i)(2-i-z)}{2(z-1-i)}$$

$$\therefore \frac{w}{i} = \frac{2(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$

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