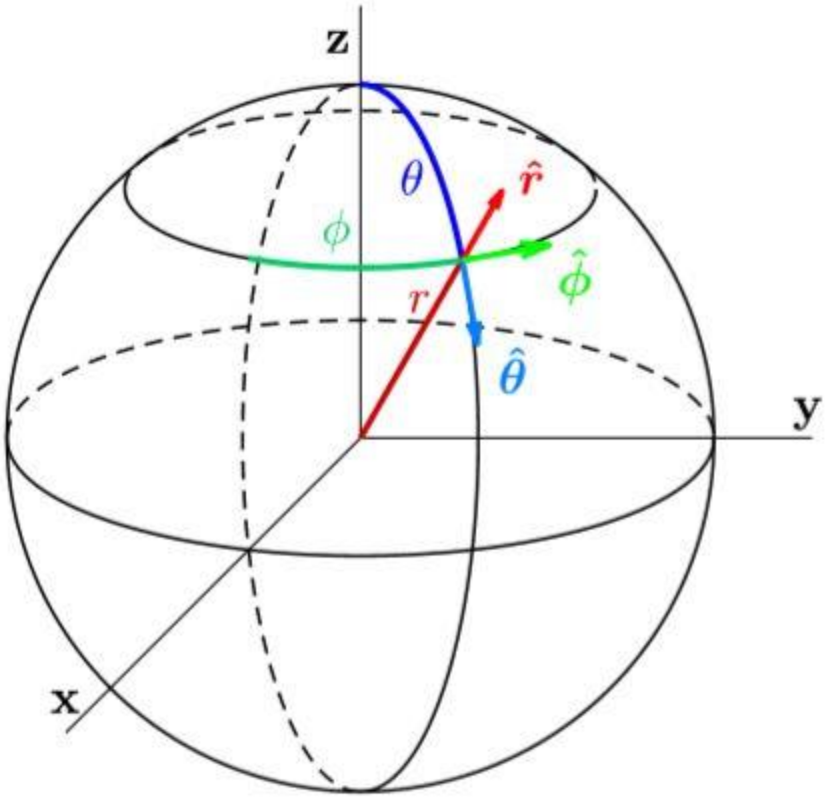
ENGINEERING MATHEMATICS – II

UNIT-2 VECTOR CALCULUS



SYLLABUS

- Gradient and directional derivative - Divergence and curl
- Vector identities – Irrotational and Solenoid vector fields
- Line integral over a plane curve – Surface integral – Area of a curved surface
- Volume integral – Green's, Gauss divergence and Stoke's theorems
- Verification and application in evaluating line, surface and volume integrals

BASIC DEFINITIONS AND FORMULAE

Definition: Gradient

Let $\phi(x,y,z)$ be a scalar function which is continuously differentiable, then the gradient of ϕ is denoted by $grad\ \phi$ or $\nabla\phi$ and is defined by

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right)\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \right)$$

Directional Derivative

The component of $\nabla\phi$ in the direction of the vector a is given by

$$\text{Directional derivative} = \frac{\nabla\phi . a}{|a|}.$$

FORMULA BASED ON GRADIENT

Formula

(i) $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right)\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \right)$

(ii) Directional derivative = $\frac{\nabla\phi . a}{|a|}$.

(iii) A unit normal to the surface $\phi(x, y, z) = c$ is $\frac{\nabla\phi}{|\nabla\phi|}$

(iv) Angle between two surfaces $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ is given by

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

(v) Two surfaces are $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ orthogonal if $\nabla\phi_1 . \nabla\phi_2 = 0$.

SIMPLE PROBLEMS

PROBLEM - 1

Find $\nabla\phi$ **and** $|\nabla\phi|$ **if** $\phi = 2xz^4 - x^2 y$ **at (2,-2,1)**

Solution: $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(2xz^4 - x^2 y\right) = \left((2z^4 - 2xy)\vec{i} - x^2\vec{j} + 8xz^3\vec{k}\right)$

$[\nabla\phi]_{(2,-2,1)} = 10\vec{i} - 4\vec{j} + 16\vec{k}$ and $|\nabla\phi| = \sqrt{(10)^2 + (-4)^2 + (16)^2} = \sqrt{100 + 16 + 256} = \sqrt{372} = 2\sqrt{93}$.

PROBLEM - 2

Find the unit normal vector to the surface $\phi = x^2 + y^2 - z$ at (1,-2,5)

Solution: $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[x^2 + y^2 - z\right] = (2x)\vec{i} + (2y)\vec{j} + (-1)\vec{k}$

$[\nabla\phi]_{(1,-2,5)} = 2\vec{i} - 4\vec{j} - \vec{k}$

Unit normal to the surface $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{2^2 + (-4)^2 + (-1)^2}} = \dfrac{2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$.

PROBLEM - 3

Find the directional derivative of $\phi = xyz$ at (1, 1, 1) in the direction of $\vec{i} + \vec{j} + \vec{k}$.

Solution: $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz) = (yz)\vec{i} + (xz)\vec{j} + (xy)\vec{k}$. $[\nabla\phi]_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$

The directional derivative of ϕ in the direction of $\vec{i} + \vec{j} + \vec{k} = \nabla\phi \bullet \frac{\vec{a}}{|\vec{a}|} = \nabla\phi \bullet \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{1^2 + 1^2 + 1^2}}$

$$= (\vec{i} + \vec{j} + \vec{k}) \bullet \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

PROBLEM -4

Find the Directional derivative of $\phi = 4xz^2 + x^2yz$ at (1, -2,1) in the direction $2\vec{i} + 3\vec{j} + 4\vec{k}$. (N / D2016)

Solution: Given $\phi = 4xz^2 + x^2yz$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[4xz^2 + x^2yz\right] = (4z^2 + 2xyz)\vec{i} + (x^2z)\vec{j} + (8xz + x^2y)\vec{k}$$

$$[\nabla\phi]_{(1,-2,1)} = (4-4)\vec{i} + \vec{j} + (8-2)\vec{k} = \vec{j} + 6\vec{k}$$

$$\hat{n} = \left(\frac{2\vec{i} + 3\vec{j} + 4\vec{k}}{\sqrt{(2)^2 + (3)^2 + (4)^2}}\right) = \left(\frac{2\vec{i} + 3\vec{j} + 4\vec{k}}{\sqrt{29}}\right)$$ where $\hat{n}$ is the unit normal vector.

Directional derivative of ϕ is

$$\nabla\phi\cdot\hat{n} = \left[(\vec{j} + 6\vec{k})\cdot\left(\frac{2\vec{i} + 3\vec{j} + 4\vec{k}}{\sqrt{29}}\right)\right] = \frac{2(0) + 3(1) + 6(4)}{\sqrt{29}} = \frac{27}{\sqrt{29}}$$

PROBLEM -5

Find the directional derivative of $\phi = xy^2z^3$ at (1,1,1) along normal to the surface $x^2+xy+z^2=3$ at the point (1,1,1)

Solution: Given $\phi = xy^2z^3$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left[xy^2z^3\right] = (y^2z^3)\vec{i}+(2xyz^3)\vec{j}+(3xy^2z^2)\vec{k}$$

$$[\nabla\phi]_{(1,1,1)} = \vec{i}+2\vec{j}+3\vec{k}$$

Given $\varphi = x^2+xy+z^2-3$

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left[x^2+xy+z^2-3\right] = (2x+y)\vec{i}+(x)\vec{j}+(2z)\vec{k}$$

$$[\nabla\varphi]_{(1,1,1)} = 3\vec{i}+\vec{j}+2\vec{k}$$

Unit normal to the surface $= \vec{a} = \dfrac{\nabla\varphi}{|\nabla\varphi|} = \dfrac{3\vec{i}+\vec{j}+2\vec{k}}{\sqrt{9+1+4}} = \dfrac{3\vec{i}+\vec{j}+2\vec{k}}{\sqrt{14}}$

$$\nabla\phi\cdot\vec{a} = \left[(\vec{i}+2\vec{j}+3\vec{k})\cdot\left(\frac{3\vec{i}+\vec{j}+2\vec{k}}{\sqrt{14}}\right)\right] = \frac{3+2+6}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

PROBLEM -6

In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find the magnitude of this maximum.

Solution: Given $\phi = x^2y^2z^4$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[x^2y^2z^4\right] = (2xy^2z^4)\vec{i} + (2yx^2z^4)\vec{j} + (4z^3x^2y^2)\vec{k}$$

$$\left[\nabla\phi\right]_{(3,1,-2)} = 96\vec{i} + 288\vec{j} - 288k$$

∴ The maximum directional derivative occurs in the direction of

$$\nabla\phi = 96(\vec{i} + 3\vec{j} - 3\vec{k})$$

The magnitude of this maximum directional derivative is $\left|\nabla\phi\right| = 96\sqrt{1+9+9} = 96\sqrt{19}$.

PROBLEM – 7

Find the values of constants *a, b, c* so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a magnitude 64 in the direction parallel to *z*-axis. (N /D2015)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) + \vec{j}\,\frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) + \vec{k}\,\frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\vec{i} + (2axy + bz)\vec{j} + (by + 2czx^3)\vec{k}$$

at the point $(1,2,-1)$

$$\nabla\phi = \vec{i}(4a+3c) + \vec{j}(4a-b) + \vec{k}(2b-2c) \quad \rightarrow (1)$$

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The Directional Derivative is Maximum in the direction of $\nabla\phi$ i.e. in the direction of $\vec{i}(4a+3c)+\vec{j}(4a-b)+\vec{k}(2b-2c)$. But it is given that directional derivative is maximum in the direction of z-axis i.e., in the direction of $0\vec{i}+0\vec{j}+\vec{k}$. Therefore, $\nabla\phi$ and z-axis are parallel.

$$\frac{4a+3c}{0}=\frac{4a-b}{0}=\frac{2b-2c}{1}=l \text{, (say)}$$

$$4a+3c=0 \quad \rightarrow (2)$$

$$4a-b=0 \quad \rightarrow (3)$$

Cont...

substituting in eq.(1),

$\nabla\phi = (2b - 2c)\vec{k}$

Maximum value of directional derivative is $|\nabla\phi|$. But it is given as 64.

$|\nabla\phi| = 64$

$|(2b - 2c)\vec{k}| = 64$

$2b - 2c = 64$, $b - c = 32$

From eq (2) & (3)

$4a + 3c = 0$, $4a - b = 0$, Solving, $b = -3c$

Substituting in $b - c = 32$, $-4c = 32$

$a = 6$, $b = 24$, $c = -8$.

PROBLEM - 8

Find the directional derivative of $\phi = x^2 + y^2 - 2z^2$ at P (1, 0, 2) in the direction of the line PQ where Q is the point (2, 3, 4).

Solution: $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 - 2z^2\right) = (2x)\vec{i} + (2y)\vec{j} - (4z)\vec{k}$

$[\nabla\phi]_{(1,0,2)} = 2\vec{i} - 8\vec{k}$

Position Vector of $Q = 2\vec{i} + 3\vec{j} + 4\vec{k}$ andPosition Vector of $P = \vec{i} + 2\vec{k}$

$\overline{PQ} = (Position Vector\ of\ Q) - (Position Vector\ of\ P) = \left(2\vec{i} + 3\vec{j} + 4\vec{k}\right) - \left(\vec{i} + 2\vec{k}\right) = \vec{i} + 3\vec{j} + 2\vec{k}$

The directional derivative of ϕ in the direction of $\vec{i} + 3\vec{j} + 2\vec{k} = \nabla\phi \bullet \frac{\vec{a}}{|\vec{a}|} = \nabla\phi \bullet \frac{(\vec{i} + 3\vec{j} + 2\vec{k})}{\sqrt{1^2 + 3^2 + 2^2}}$

$= (2\vec{i} - 8\vec{k}) \bullet \frac{(\vec{i} + 3\vec{j} + 2\vec{k})}{\sqrt{14}} = \frac{2 - 16}{\sqrt{14}} = -\sqrt{14}$

PROBLEM - 9

Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1, 1, 1)

Solution: Let $\phi_1 = y^2 - x\log z - 1$

$\nabla\phi_1 = -\log z\,\vec{i} + 2y\,\vec{j} - \frac{x}{z}\vec{k}$, $(\nabla\phi_1)(1,1,1) = 2\vec{j} - \vec{k}$ and $|\nabla\phi_1| = \sqrt{5}$

Let $\phi_2 = x^2 y - 2 + z$

$\nabla\phi_2 = (2xy)\,\vec{i} + x^2\,\vec{j} + (1)\vec{k}$, $(\nabla\phi_2)(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$ and $|\nabla\phi_2| = \sqrt{6}$

$$cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(2\vec{j} - \vec{k}\right).\left(2\vec{i} + \vec{j} + \vec{k}\right)}{\left(\sqrt{5}\right)\left(\sqrt{6}\right)} = \frac{0 + 2 - 1}{\sqrt{30}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

PROBLEM - 10

Find the angle between normal to the surface $xy = z^2$ at the point (−2, −2,2) and (1,9, −3).

Solution: Let $\phi = xy - z^2$

$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$,

At the point $(-2,-2,2) \Rightarrow [\nabla\phi_1]_{(-2,-2,2)} = -2\vec{i} - 2\vec{j} - 4\vec{k}$ *and* $|\nabla\phi_1| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

At the point $(1,9,-3) \Rightarrow [\nabla\phi_2]_{(1,9,-3)} = 9\vec{i} + \vec{j} + 6\vec{k}$ *and* $|\nabla\phi_2| = \sqrt{81+1+36} = \sqrt{118}$

$$cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(-2\vec{i} - 2\vec{j} - 4\vec{k}\right).\left(9\vec{i} + \vec{j} + 6\vec{k}\right)}{2\sqrt{6}\left(\sqrt{118}\right)} = \frac{-18-2-24}{2\sqrt{708}} = -\frac{-11}{\sqrt{177}}$$

$$\theta = cos^{-1}\left(-\frac{11}{\sqrt{177}}\right).$$

PROBLEM - 11

Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point (2,-1,2)

(A / M2017)

Solution: Let $\phi_1 = x^2+y^2+z^2-9$

$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$, $(\nabla\phi_1)(2,-1,2) = 4\vec{i}-2\vec{j}+4\vec{k}$ and $|\nabla\phi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$

Let $\phi_2 = z-x^2-y^2+3$

$\nabla\phi_2 = (-2x)\vec{i}-2y\vec{j}+(1)\vec{k}$, $(\nabla\phi_2)(2,-1,2) = -4\vec{i}+2\vec{j}+\vec{k}$ and $|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$

$$cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(4\vec{i}-2\vec{j}+4\vec{k}\right).\left(-4\vec{i}+2\vec{j}+\vec{k}\right)}{6\left(\sqrt{21}\right)} = \frac{-16-4+4}{6\sqrt{21}} = -\frac{8}{3\sqrt{21}}$$

$$\Rightarrow \quad \theta = cos^{-1}\left(-\frac{8}{3\sqrt{21}}\right).$$

SCALAR POTENTIAL
PROBLEM - 12

If $\nabla\phi=(y^2-2xyz^3)\vec{i}+(3+2xy-x^2z^3)\vec{j}+(6z^3-3x^2yz^2)\vec{k}$, find ϕ.

Solution: $\nabla\phi=\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)=(y^2-2xyz^3)\vec{i}+(3+2xy-x^2z^3)\vec{j}+(6z^3-3x^2yz^2)\vec{k}$

Equating the components of $\vec{i},\vec{j},\vec{k}$,

$\frac{\partial\phi}{\partial x}=(y^2-2xyz^3)$...(1)	$\frac{\partial\phi}{\partial y}=(3+2xy-x^2z^3)$...(2)	$\frac{\partial\phi}{\partial z}=(6z^3-3x^2yz^2)$...(3)

Integrating (1) partially w.r.t. x, we get $\phi=xy^2-x^2yz^3+\ f_1(y,z)$ ...(4)

Integrating (2) partially w.r.t. y, we get $\phi=3y+xy^2-x^2yz^3+f_2(x,z)$ ...(5)

Integrating (3) partially w.r.t. z, we get $\phi=\frac{3}{2}z^4-x^2yz^3+f_3(x,y)$ ...(6)

From (4), (5) and (6), collecting non repeating terms alone, we get $\phi=3y+xy^2-x^2yz^3+\frac{3}{2}z^4+c$

PROBLEM -13

Find 'a' and 'b' so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ cut orthogonally at $(2, -1, -3)$.

Solution

Let $\phi_1 = ax^3 - by^2z - (a+3)x^2$ $\qquad\qquad$ $\phi_2 = 4x^2y - z^3 - 11$

$\nabla\phi_1 = [3ax^2 - (a+3)2x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$ $\qquad$ $\nabla\phi_2 = 8xy\vec{i} - 4x^2\vec{j} - 3z^2\vec{k}$

$(\nabla\phi_1)_{(2,-1,-3)} = (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}$ $\qquad$ $(\nabla\phi_2)_{(2,-1,-3)} = -16\vec{i} - 16\vec{j} - 27\vec{k}$

Since the surfaces cut orthogonally $\Rightarrow \nabla\phi_1 . \nabla\phi_2 = 0$

$\Rightarrow -16(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b = 192 \qquad \rightarrow (1)$

Since the points $(2, -1, -3)$ lies on the surface $\phi(x, y, z) = 0$, we have

$8a + 3b - 4a = 12$

$\Rightarrow 4a + 3b = 12 \qquad \rightarrow (2)$

Solving $(1) \& (2)$ we get $a = -2.333 \quad b = 7.111$

PROBLEM -14

If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, Find $\phi(x,y,z)$ given that $\phi(1,-2,2) = 4$

(M / J2016)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z} \quad \rightarrow (1)$$

$$Given\ \nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \quad \rightarrow (2)$$

$$\text{Given } \nabla\phi = \left(\vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}\right) = (2xyz^3)\vec{i} + (x^2z^3)\vec{j} + (3x^2yz^2)\vec{k}$$

Equating the components of $\vec{i}, \vec{j}, \vec{k}$,

Cont...

$$\frac{\partial \phi}{\partial x} = 2xyz^3 \qquad ---(3)$$

$$\frac{\partial \phi}{\partial y} = x^2 z^3 \qquad ---(4)$$

$$\frac{\partial \phi}{\partial z} = 3x^2 yz^2 \qquad ---(5)$$

Integrating (3) *w.r.t* x (keeping y and z constant)

we get $\phi = x^2yz^3 + f_1(y, z)$ .....(I)

Integrating (4) *w.r.t* y (keeping x and z constant)

we get $\phi = x^2yz^3 + f_2(x, z)$ .....(II)

Integrating (5) *w.r.t* z (keeping x and y constant)

$\phi = x^2yz^3 + f_3(x, y)$ ......(III)

From (I), (II) and (III), collecting non repeating terms alone, we get $\phi = x^2yz^3 + c$

Given $\phi\ (1, -2, 2) = 4$

$\Rightarrow (1)^2(-2)(2)^3 + c = 4 \Rightarrow -16 + c = 4$

$\Rightarrow c = 4 + 16 = 20$

$\therefore\ \phi = x^2yz^3 + 20$

PROBLEM - 15

If $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$, Find $\phi(x, y, z)$

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \quad \rightarrow (1)$$

$$Given\ \nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k} \quad \rightarrow (2)$$

$\therefore$ *comparing* (1) & (2)

$$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3 \quad \rightarrow (3)$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3 \quad \rightarrow (4)$$

$$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2 \quad \rightarrow (5)$$

Cont...

Integrating (3) *w.r.t* x (keeping y and z constant)

$$\phi = y^2x - \frac{2x^2yz^3}{2} + f_1(y,z) = y^2x - x^2yz^3 + f_1(y,z) \to (i)$$

Integrating (4) *w.r.t* y (keeping x and z constant)

$$\phi = 3y + \frac{2xy^2}{2} - x^2z^3y + f_2(x,z) = 3y + xy^2 - x^2z^3y + f_2(x,z) \to (ii)$$

Integrating (5) *w.r.t* z (keeping x and y constant)

$$\phi = \frac{6z^4}{4} - \frac{3x^2yz^3}{3} + f_3(x,y) = \frac{3z^4}{2} - x^2yz^3 + f_3(x,y) \to (iii)$$

From (i), (ii) *and* (iii), *collecting non repeating terms alone*

$$\therefore \quad \phi = 3y + xy^2 - x^2z^3y + \frac{3z^4}{2} + c \text{ where } c \text{ is a constant}$$

PROBLEM -16

Show that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is a conservative force field and hence find its scalar potential. (N / D2014)

Solution:

Given $\vec{F} = \left(y^2 \cos x + z^3\right)\vec{i} + \left(2y\sin x - 4\right)\vec{j} + 3xz^2\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$\nabla \times \vec{F} = = \vec{i}[0-0] - \vec{j}\left[3z^2 - 3z^2\right] + \vec{k}[2y\cos x - 2y\cos x] = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

Hence $\vec{F}$ is irrotational

Cont...

Finding Scalar Potential

$$\vec{F} = \nabla\phi$$

$$\Rightarrow \left(y^2 \cos x + z^3\right)\vec{i} + \left(2y\sin x - 4\right)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$$

Equating the coefficient $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \int \partial\phi = \int (y^2\cos x + z^3)dx$$

$$\Rightarrow \phi_1 = y^2 \sin x + z^3 x + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \int \partial\phi = \int \left(2y\sin x - 4\right)dy$$

Cont...

$$\Rightarrow \phi_2 = 2(\sin x)\frac{y^2}{2} - 4y + f_2(x,z) = y^2 \sin x - 4y + f_2(x,z)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \Rightarrow \int \partial \phi = \int 3xz^2 dz$$

$$\Rightarrow \phi_3 = 3x\frac{z^3}{3} + f_3(x,y) = xz^3 + f_3(x,y)$$

From (4), (5) and (6), collecting non repeating terms alone, we get

$$\therefore \phi = y^2 \sin x + xz^3 - 4y + C$$

DIVERGENCE AND CURL

Definition: Divergence

Let $F = F_1\, i + F_2\, j + F_3\, k$ be a vector point function. Then divergence of F is denoted by

$div\, F$ or $\nabla . F$ and is defined by $\nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Definition: Curl

Let $F = F_1\, i + F_2\, j + F_3\, k$ be a vector point function. Then curl of F is denoted by

$$curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

SOLENOIDAL AND IRROTATIONAL

Definition: Solenoidal

A vector function F is said to be solenoidal if $\nabla \bullet F = 0$

Definition: Irrotational

A vector function F is said to be irrotational if $\nabla \times F = 0$

Definition: Conservative field and Scalar potential

- $\nabla X F = 0$, Then F is called Conservative field.
- If F is irrotational, then a scalar function ϕ can be found so that $F = \nabla\phi$ and ϕ is called scalar potential of F

Problem 1

If $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution: A vector function $\vec{V}$ is said to be solenoidal if $\nabla \bullet \vec{V} = 0$

Given $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$ is solenoidal.

$$\left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} \right) . ((x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}) = 0$$

$$\left(\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+2\lambda z) \right) = 0 \Rightarrow 1+1+2\lambda = 0 \Rightarrow \lambda = -1$$

Problem 2

Show that a vector field $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ **is irrotational.**

Solution: $curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2 - y^2 + x) & -(2xy + y) & 0 \end{vmatrix}$

$$= \vec{i}\left[0 + \frac{\partial}{\partial z}(2xy + y)\right] - \vec{j}\left[0 - \frac{\partial}{\partial z}(x^2 - y^2 + x)\right] + \vec{k}\left[-\frac{\partial}{\partial x}(2xy + y) - \frac{\partial}{\partial y}(x^2 - y^2 + x)\right]$$

$$= \vec{i}\,0 + \vec{j}\,0 + \vec{k}\left[-2y + 2y\right] = 0 \;\therefore\; \vec{F} \text{ is irrotational.}$$

Problem 3

For what values of 'a', 'b' and 'c' such that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **is irrotational.**

Solution: If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = \vec{0}$$

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$. $\Rightarrow a = 4$; $b = 2$; $c = -1$

Problem 4

If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, **find div(curl** $\vec{F}$ **)**

Solution: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$

$$= \vec{i}\left(\left(\frac{\partial}{\partial y}\right)(z^3) - \left(\frac{\partial}{\partial z}\right)(y^3)\right) - \vec{j}\left(\left(\frac{\partial}{\partial x}\right)(z^3) - \left(\frac{\partial}{\partial z}\right)(x^3)\right) + \vec{k}\left(\left(\frac{\partial}{\partial x}\right)(y^3) - \left(\frac{\partial}{\partial y}\right)(x^3)\right) = 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\text{div}\,(\text{curl}\,\vec{F}) = \nabla \bullet \left(\nabla \times \vec{F}\right) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).(0\vec{i} - 0\vec{j} + 0\vec{k}) = 0$$

Problem 5

For what values of 'a', 'b' and 'c' such that $\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ is irrotational. Find its scalar potential (A /M 17,18)

Solution: If $\vec{F}$ is irrotational then $\nabla\times\vec{F}=\vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}=\vec{0}$$

$\vec{i}\,(c+1)-\vec{j}\,(4-a)+\vec{k}\,(b-2)=0\vec{i}+0\vec{j}+0\vec{k}$, $\Rightarrow a = 4\ ;\ b=2\ ;\ c = -1$

Finding Scalar Potential

$\vec{F}=\nabla\phi$

$$\Rightarrow(x+2y+4z)\vec{i}+(2x-3y-z)\vec{j}+(4x-y+2z)\vec{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}$$

Cont...

Equating the coefficient $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z \Rightarrow \int \partial \phi = \int (x + 2y + 4z)dx \Rightarrow \phi_1 = \frac{x^2}{2} + 2xy + 4zx + f_1(y, z) \rightarrow (4)$$

$$\frac{\partial \phi}{\partial y} = (2x - 3y - z) \Rightarrow \int \partial \phi = \int (2x - 3y - z)\, dy \Rightarrow \phi_2 = 2xy - \frac{3y^2}{2} - zy + f_2(x, z) \rightarrow (5)$$

$$\frac{\partial \phi}{\partial z} = (4x - y + 2z) \Rightarrow \int \partial \phi = \int (4x - y + 2z)\, dz \Rightarrow \phi_3 = 4xz - zy + z^2 + f_3(x, y) \rightarrow (6)$$

From (4), (5) and (6), collecting non repeating terms alone, we get

$$\therefore \phi = \frac{x^2}{2} + 2xy + 4xz - zy - \frac{3y^2}{2} + z^2 + C$$

Problem 6

Prove that div $\vec{r}$ **=3 and curl** $\vec{r} = \vec{0}$ **.**

Solution: $\operatorname{div}\vec{r} = \nabla \bullet \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right) \bullet \left(x\vec{i} + y\vec{j} + z\vec{k} \right) = 1 + 1 + 1 = 3$

$$\operatorname{curl}\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

PROBLEM 7

Evaluate $\nabla^2(\log r)$

Solution: $\nabla^2(\log r) = \sum \frac{\partial^2}{\partial x^2}[\log r] = \sum \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\log r\right] = \sum \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{\partial r}{\partial x}\right] = \sum \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{x}{r}\right] = \sum \frac{\partial}{\partial x}\left[\frac{x}{r^2}\right]$

$$= \frac{3r^2(1) - \left((x)(2r)\left(\frac{x}{r}\right) + (y)(2r)\left(\frac{y}{r}\right) + (z)(2r)\left(\frac{z}{r}\right)\right)}{r^4} = \frac{3r^2 - 2x^2 - 2y^2 - 2z^2}{r^4} = \frac{3r^2 - 2r^2}{r^4} = \frac{1}{r^2}$$

Problem 8

Find the value of 'n' so that the vector $r^n\vec{r}$ is both irrotational and solenoidal.

Solution:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |r| = \sqrt{x^2 + y^2 + z^2}$

$r^n = \left(x^2 + y^2 + z^2\right)^{n/2}$

$r^n\vec{r} = r^n\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

$$\nabla \times \left(r^n\vec{r}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} \quad \because\ r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r};\quad \frac{\partial r}{\partial y} = \frac{y}{r};\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

CONT...

$$= \vec{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}\left(r^n y\right)\right) - \vec{j}\left(\frac{\partial}{\partial x}\left(r^n z\right) - \frac{\partial}{\partial z}\left(r^n x\right)\right) + \vec{k}\left(\frac{\partial}{\partial x}\left(r^n y\right) - \frac{\partial}{\partial y}\left(r^n x\right)\right)$$

$$= \vec{i}\left(znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right) - \vec{j}\left(znr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial z}\right) + \vec{k}\left(ynr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left(znr^{n-1}\frac{y}{r} - ynr^{n-1}\frac{z}{r}\right) - \vec{j}\left(znr^{n-1}\frac{x}{r} - xnr^{n-1}\frac{z}{r}\right) + \vec{k}\left(ynr^{n-1}\frac{x}{r} - xnr^{n-1}\frac{y}{r}\right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

$\therefore r^n\vec{r}$ Is irrotational for all values of n.

Cont...

$$\nabla.(r^n\vec{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(r^n\left(x\vec{i} + y\vec{j} + z\vec{k}\right)\right) = \frac{\partial}{\partial x}\left(r^n x\right) + \frac{\partial}{\partial y}\left(r^n y\right) + \frac{\partial}{\partial z}\left(r^n z\right)$$

$$= r^n + xnr^{n-1}\frac{\partial r}{\partial x} + r^n + ynr^{n-1}\frac{\partial r}{\partial y} + r^n + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-2}\left(x^2 + y^2 + z^2\right) = 3r^n + nr^{n-2}\left(r^2\right) = 3r^n + nr^n = (3+n)r^n$$

$r^n\vec{r}$ is Solenoidal $\Rightarrow \nabla.r^n\vec{r} = 0$

$(3+n)r^n = 0 \Rightarrow n = -3.$ Therefore $r^n\vec{r}$ is Solenoidal when $n = -3.$

Problem 9

Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$ and hence find the value of $\nabla^2\left(\frac{1}{r}\right)$.

Solution:

$$\nabla\left(r^n\right) = \sum \frac{\partial}{\partial x}\left[r^n\right] = \sum\left[nr^{n-1}\frac{\partial r}{\partial x}\right] = \sum\left[nr^{n-1}\frac{x}{r}\right]$$

$$= \vec{i}\ \ nr^{n-1}\left(\frac{x}{r}\right) + \vec{j}\ \ nr^{n-1}\left(\frac{y}{r}\right) + \vec{k}\ \ nr^{n-1}\left(\frac{z}{r}\right)$$

$$= \vec{i}\, nr^{n-2}x + \vec{j}\, nr^{n-2}y + \vec{k}\, nr^{n-2}z$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})\ (\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \nabla\left(r^n\right) = nr^{n-2}\vec{r}.$$

CONT…

Now

$$\nabla^2\left(r^n\right)=\nabla.\nabla\left(r^n\right)=\nabla.\left(nr^{n-2}\vec{r}\right)=\left(\vec{i}\ \frac{\partial}{\partial x}+\vec{j}\ \frac{\partial}{\partial y}+\vec{k}\ \frac{\partial}{\partial z}\right)nr^{n-2}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$=\left(\vec{i}\ \frac{\partial}{\partial x}(nr^{n-2}x)+\vec{j}\ \frac{\partial}{\partial y}(nr^{n-2}y)+\vec{k}\ \frac{\partial}{\partial z}(nr^{n-2}z)\right)$$

$$=nr^{n-2}(1)+xn(n-2)r^{n-3}\frac{\partial r}{\partial x}+nr^{n-2}(1)+yn(n-2)r^{n-3}\frac{\partial r}{\partial y}+nr^{n-2}(1)+zn(n-2)r^{n-3}\frac{\partial r}{\partial z}$$

$$=3nr^{n-2}+n(n-2)r^{n-3}\left[\frac{x^2+y^2+z^2}{r}\right]=3nr^{n-2}+n(n-2)r^{n-4}(x^2+y^2+z^2)$$

$$=3nr^{n-2}+n(n-2)r^{n-4}(r^2)=3nr^{n-2}+n(n-2)r^{n-2}=r^{n-2}(n^2+3n-2n)$$

$$=n(n+1)r^{n-2}$$

Finding $\nabla^2\left(\frac{1}{r}\right)$

Taking $n=-1$ in above step, we get $\nabla^2\left(\frac{1}{r}\right)=0.$

Problem 10

Prove that $\nabla^2(f(r)) = f''(r) + \frac{2}{r} f'(r)$ and find $f(r)$ such that $\nabla^2(f(r)) = 0$. (Or)

Prove that $\nabla^2(f(r)) = \frac{d^2}{dr^2}(f(r)) + \frac{2}{r}\frac{d}{dr}(f(r))$ and find $f(r)$ suchthat $\nabla^2(f(r)) = 0$.

Solution:

$$\nabla(f(r)) = \sum \frac{\partial}{\partial x}[f(r)] = \sum\left[f'(r)\frac{\partial r}{\partial x}\right] = \sum\left[f'(r)\frac{x}{r}\right]$$

$$= \vec{i}\ f'(r)\left(\frac{x}{r}\right) + \vec{j} f'(r)\left(\frac{y}{r}\right) + \vec{k}\ f'(r)\left(\frac{z}{r}\right)$$

$$= f'(r)\frac{\left(x\vec{i} + y\vec{j} + z\vec{k}\right)}{r}$$

Cont...

$$\nabla^2(f(r)) = \nabla.\nabla(f(r)) = \nabla.\left(f'(r)\frac{(x\vec{i}+y\vec{j}+z\vec{k})}{r}\right) = \left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right)f'(r)\frac{(x\vec{i}+y\vec{j}+z\vec{k})}{r}$$

$$= \left(\vec{i}\,\frac{\partial}{\partial x}\left(f'(r)\left(\frac{x}{r}\right)\right)+\vec{j}\,\frac{\partial}{\partial y}\left(f'(r)\left(\frac{y}{r}\right)\right)+\vec{k}\,\frac{\partial}{\partial z}\left(f'(r)\left(\frac{z}{r}\right)\right)\right)$$

$$\sum\frac{\partial}{\partial x}\left(f'(r)\left(\frac{x}{r}\right)\right) = \left(\frac{x}{r}\right)f''(r)\frac{\partial r}{\partial x}+f'(r)\left[\frac{r.1-x\frac{\partial r}{\partial x}}{r^2}\right] = f''(r)\left(\frac{x}{r}\right)^2+f'(r)\left[\frac{r^2-x^2}{r^3}\right]$$

Cont...

$$\nabla^2(f(r)) = \frac{\partial}{\partial x}\left(f'(r)\left(\frac{x}{r}\right)\right) + \frac{\partial}{\partial y}\left(f'(r)\left(\frac{y}{r}\right)\right) + \frac{\partial}{\partial z}\left(f'(r)\left(\frac{z}{r}\right)\right)$$

$$= f''(r)\left(\frac{x}{r}\right)^2 + f'(r)\left[\frac{r^2 - x^2}{r^3}\right] + f''(r)\left(\frac{y}{r}\right)^2 + f'(r)\left[\frac{r^2 - y^2}{r^3}\right] + f''(r)\left(\frac{z}{r}\right)^2 + f'(r)\left[\frac{r^2 - z^2}{r^3}\right]$$

$$\nabla^2(f(r)) = \frac{f''(r)}{r^2}\left[x^2 + y^2 + z^2\right] + f'(r)\left[\frac{3r^2 - (x^2 + y^2 + z^2)}{r^3}\right] = \frac{f''(r)}{r^2}\left[r^2\right] + f'(r)\left[\frac{3r^2 - r^2}{r^3}\right]$$

$$\nabla^2(f(r)) = f''(r) + \frac{2}{r} f'(r)$$

Cont...

Finding $f(r)$

Suppose $\nabla^2 (f(r)) = 0 \Rightarrow f''(r) + \frac{2}{r} f'(r) = 0$

$$\Rightarrow f''(r) = -\frac{2}{r} f'(r) \quad \Rightarrow \frac{f''(r)}{f'(r)} = -\frac{2}{r}$$

Integrating w.r.t 'r' ,we get

$$\int \frac{f''(r)}{f'(r)}\, dr = -\int \frac{2}{r}\, dr$$

$$\Rightarrow \log\left[f'(r)\right] = -2\log r + \log c = \log r^{-2} + c = \log\left(\frac{1}{r^2}\right) + \log c$$

Eliminating log on both sides

$$\Rightarrow f'(r) = \left(\frac{c}{r^2}\right)$$

Cont…

Integrating w.r.t 'r' ,we get

$$\Rightarrow \int f^{'}(r)\,dr = c\int\left(\frac{1}{r^2}\right)dr = c\int r^{-2}dr$$

$$\Rightarrow f(r) = \frac{r^{-2+1}}{-2+1} = -\frac{c}{r}$$

$$\Rightarrow f(r) = -\frac{c}{r}$$

Problem 11

Prove that $Curl\ Curl\ \vec{F} = grad\ div\,\vec{F} - \nabla^2\ \vec{F}$ (M / J2016)

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$Curl\,(Curl\ \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)_1 & \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right) & \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

Cont...

$$=\sum\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]\vec{i}=\sum\left[\left(\frac{\partial^2 F_2}{\partial y\,\partial x}+\frac{\partial^2 F_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

$$=\sum\left[\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial x\,\partial y}+\frac{\partial^2 F_3}{\partial x\,\partial z}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right]\vec{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\nabla.\ \vec{F}\right)-\nabla^2 F_1\right]\vec{i}$$

$$=\left[\vec{i}\,\frac{\partial}{\partial x}\left(\nabla.\ \vec{F}\right)+\vec{j}\,\frac{\partial}{\partial y}\left(\nabla.\ \vec{F}\right)+\vec{k}\,\frac{\partial}{\partial z}\left(\nabla.\ \vec{F}\right)\right]-\nabla^2\left[F_1\vec{i}+F_2\vec{j}+F_3\vec{k}\right]$$

$$curl(curl\vec{F})=\nabla(\nabla.\ \vec{F})-\nabla^2\vec{F}$$

PROBLEM 12

Show that $\vec{F} = yz^2\vec{i} + (xz^2 - 1)\vec{j} + (2xyz - 2)\vec{k}$ is irrotational and hence find its scalar potential. (N / D2019)

Given $\vec{F} = yz^2\vec{i} + (xz^2 - 1)\vec{j} + (2xyz - 2)\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & (xz^2 - 1) & (2xyz - 2) \end{vmatrix}$$

$$\nabla \times \vec{F} = = \vec{i}[2xz - 2xz] - \vec{j}[2yz - 2yz] + \vec{k}\left[z^2 - z^2\right] = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

Cont…

Finding Scalar Potential

$$\vec{F} = \nabla\phi$$

$$\Rightarrow yz^2\vec{i} + \left(xz^2 - 1\right)\vec{j} + \left(2xyz - 2\right)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$$

Equating the coefficient $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \int\partial\phi = \int(yz^2)dx \Rightarrow \phi_1 = xyz^2 + f_1(y, z) \rightarrow (4)$$

Cont...

$$\frac{\partial \phi}{\partial y} = \left(xz^2 - 1\right) \Rightarrow \int \partial \phi = \int \left(xz^2 - 1\right) dy \Rightarrow \phi_2 = xyz^2 - y + f_2(x, z) \rightarrow (5)$$

$$\frac{\partial \phi}{\partial z} = \left(2xyz - 2\right) \Rightarrow \int \partial \phi = \int \left(2xyz - 2\right) dz \Rightarrow \phi_3 = xyz^2 - 2z + f_3(x, y) \rightarrow (6)$$

From (4), (5) and (6), collecting non repeating terms alone, we get

$$\therefore \phi = xyz^2 - 2z - y + C$$

Vector integration

Line integral

Problem 1

Evaluate the work done by $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ when moving a particle from x =1 and x=2 along the curve $y = x^3$.

Solution: Given $y = x^3 \Rightarrow dy = 3x^2\,dx$. $\vec{F} = 5xy\vec{i} + 2y\vec{j}$; $\vec{F}.d\vec{r} = 5xydx + 2ydy$

Work done $= \int_C \vec{F}\cdot d\vec{r} = \int_C (5x\,y\,dx + 2y\,dy) = \int_1^2 (5x^4 + 6x^5)dx = \left[x^5 + x^6\right]_1^2 = ((32+64)-(2)) = 94.$

PROBLEM 2

If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\,\vec{k}$**, then check whether the integral** $\int_C \vec{F}\cdot d\vec{r}$ **is independent of the path C.**

Solution: The line integral is independent of the path of integration if $\nabla \times \vec{F} = \vec{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2)\right)$$

$$= \vec{i}(0-0) - \vec{j}(-6x^2z + 6x^2z) + \vec{k}(4x - 4x) = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

Hence the line integral is independent of path.

Problem 3

Find $\int_C \vec{F}.d\vec{r}$ where $\vec{F}=(2y+3)\vec{i}+xz\vec{j}+(yz-x)\vec{k}$ along the line joining the points (0, 0, 0) to (2, 1, 1).

Solution

Given $\vec{F}=(2y+3)\vec{i}+xz\vec{j}+(yz-x)\vec{k}$

$d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$\vec{F}.d\vec{r}=(2y+3)dx+xz\ dy+(yz-x)dz$.

CONT...

The equation of straight line joining (0, 0, 0) to (2, 1, 1)is

$$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{1-0}\Rightarrow\frac{x}{2}=\frac{y}{1}=\frac{z}{1}=t\left(\because\ \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right)$$

$$\Rightarrow x=2t\Rightarrow dx=2dt\ ;\quad y=t\Rightarrow dy=dt\ ;\quad z=t\Rightarrow dz=dt$$

$$\int_C \overline{F}.\ \overline{dr}=\int_0^1 (2y+3)\,dx+xz\,dy+(yz-x)\,dz=\int_0^1 (2(t)+3)(2dt)+(2t.t)(dt)+(t.t-2t).dt$$

$$=\int_0^1 \left(3t^2+2t+6\right)dt=\left[3\frac{t^3}{3}+2\left(\frac{t^2}{2}\right)+6t\right]_0^1=8.$$

PROBLEM 4

If $\vec{A} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, evaluate $\int_C \vec{A}.\,d\vec{r}$, where C is the curve $y = x^2$ in the xy plane from the point (1,1) to (2,4). (N / D2019)

Solution

Given $\vec{A} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$ and $d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{A}.d\vec{r} = (5xy - 6x^2)dx + (2y - 4x)dy$

Also given that $y = x^2 \Rightarrow dy = 2x\,dx$, x varies from 1 to 2.

Cont...

$$\int_C \overline{A}.\,\overline{dr} = \int_1^2 \left(5x(x^2)-6x^2\right)dx + (2(x^2)-4x)(2xdx) = \int_1^2 (5x^3-6x^2+4x^3-8x^2)\,dx$$

$$= \int_1^2 (9x^3-14x^2)\,dx = \left[9\frac{x^4}{4}-14\frac{x^3}{3}\right]_1^2 = \left[\left(\frac{144}{4}-\frac{112}{3}\right)-\left(\frac{9}{4}-\frac{14}{3}\right)\right]$$

$$= \left[\left(\frac{135}{4}-\frac{98}{3}\right)\right] = \frac{1}{12}[405-392] = \frac{13}{12}.$$

Cont...

$$\int_C \overline{A}.\ \overline{dr} = \int_1^2 \left(5x(x^2) - 6x^2\right)dx + (2(x^2) - 4x)(2x\,dx) = \int_1^2 (5x^3 - 6x^2 + 4x^3 - 8x^2)\,dx$$

$$= \int_1^2 (9x^3 - 14x^2)\,dx = \left[9\frac{x^4}{4} - 14\frac{x^3}{3}\right]_1^2 = \left[\left(\frac{144}{4} - \frac{112}{3}\right) - \left(\frac{9}{4} - \frac{14}{3}\right)\right]$$

$$= \left[\left(\frac{135}{4} - \frac{98}{3}\right)\right] = \frac{1}{12}[405 - 392] = \frac{13}{12}.$$

Problem 5

Evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ along the curve C is $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ and t varying from -1 *to* 1.

Given $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overline{F}.d\vec{r} = xy\,dx + yz\,dy + zx\,dz$.

$x = t, \quad y = t^2, \quad z = t^3$

$dx = dt, \; dy = 2tdt, \; dz = 3t^2 dt$

and t: −1 to 1.

$$\int_C \overline{F}.\,\overline{dr} = \int_{-1}^{1} (xy)dx + yz\,dy + xz dz = \int_{-1}^{1} t^3 dt + t^5.(2t)dt + t^4(3t^2)dt = \int_{-1}^{1} t^3 dt + 2t^6 dt + 3t^6\,dt$$

$$= \int_{-1}^{1} t^3 dt + 5t^6 dt = \left[\frac{t^4}{4} + 5\frac{t^7}{7}\right]_{-1}^{1} = 0 + 5\left(\frac{1}{7} - \left(\frac{-1}{7}\right)\right) = \frac{10}{7}$$

GREEN'S THEOREM

Statement: If R is a closed region in the xy -plane bounded by a simple closed curve C and if $P(x, y)$ and $Q(x, y)$ are continuous functions of x and y having continuous partial derivatives in R,

then $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

PROBLEM 6

Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int (xdy - ydx)$, using Green's theorem.

Solution: By Green's Theorem, $\int_C P\,dx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

Take $P = -\frac{y}{2}$; $Q = \frac{x}{2} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{2}; \frac{\partial Q}{\partial x} = \frac{1}{2}$

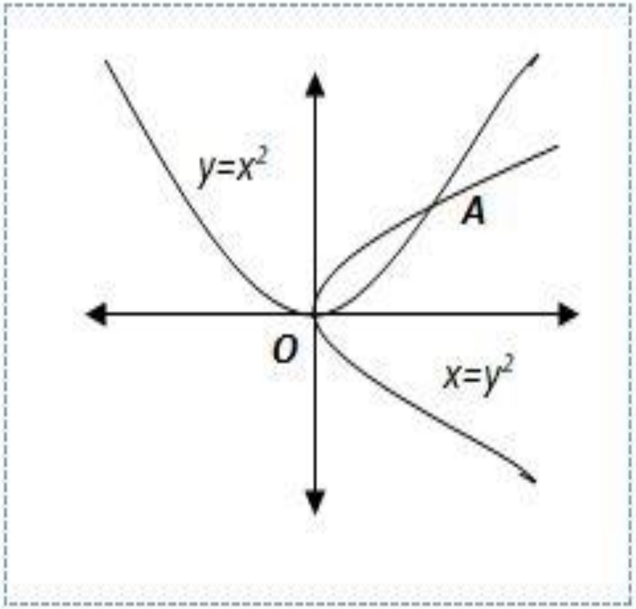
$\frac{1}{2}\int (xdy - ydx) = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dx\,dy = \iint_R dx\,dy$ = Area of the region bounded by the simple closed curve.

PROBLEM 7

Verify Green's theorem in a plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ **where** C **is the boundary of the region defined by** $x = y^2,\ y = x^2$.

Solution: By Green's theorem states that

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\right) dx\,dy$$



Given $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$$P = 3x^2 - 8y^2 \Rightarrow \frac{\partial P}{\partial y} = -16y \quad ; \quad Q = 4y - 6xy \Rightarrow \frac{\partial Q}{\partial x} = -6y$$

Evaluation of LHS:

$$\int_C (P\,dx + Q\,dy) = \int_{OA} (P\,dx + Q\,dy) + \int_{AO} (P\,dx + Q\,dy)$$

CONT…

Along OA: $y = x^2 \Rightarrow dy = 2x dx$

$$\int_{OA} P dx + Q dy = \int_{OA} (3x^2 - 8x^4) dx + (4x^2 - 6x^3) 2x dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) dx = \int_0^1 (-20x^4 + 8x^3 + 3x^2) dx$$

$$= \left[-20\frac{x^5}{5} + 8\frac{x^4}{4} + \frac{3x^3}{3} \right]_0^1 = \frac{-20}{5} + \frac{8}{5} + \frac{3}{3} = -4 + 2 + 1 = -1$$

Along AO: $y^2 = x \Rightarrow 2y dy = dx$

$$\int_{AO} P dx + Q dy = \int_{AO} (3y^4 - 8y^2) 2y\, dy + (4y - 6y^3) dy$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3) dy = \int_1^0 (6y^5 - 22y^3 + 4y) dy$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2} \right]_1^0 = \left[y^6 - \frac{11}{2} y^4 + 2y^2 \right]_1^0 = \frac{5}{2}$$

$$\therefore \int_C P dx + Q dy = -1 + \frac{5}{2} = \frac{3}{2}$$

Cont...

Evaluation of RHS:

$$\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (-6y + 16y)\, dxdy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\, dx\,dy = \int_0^1 [10xy]_{x=y}^{x=\sqrt{y}}\, dy = \int_0^1 10y\left(\sqrt{y} - y^2\right) dy$$

$$= 10\int_0^1 \left(y^{\frac{3}{2}} - y^3\right) dy = 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1 = 10\left[\frac{2}{5} - \frac{1}{4}\right] = \frac{3}{2}$$

Hence $\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\right) dxdy$

Hence Green's theorem is verified.

PROBLEM 8

Verify Green's theorem for $\int_C \left[x^2(1+y)dx + (x^3+y^3)dy \right]$ where C is the boundary of the region defined by the lines x = ±1 and y = ±1. (M / J2016)

Given $\int_c x^2(1+y)dx + (y^3+x^3)dy$

$P = x^2(1+y)$ $\qquad\qquad$ $Q = y^3 + x^3$

$\frac{\partial P}{\partial y} = x^2$ $\qquad\qquad$ $\frac{\partial Q}{\partial x} = 3x^2$

By Green's theorem $\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$

Consider $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{-1}^{1}\int_{-1}^{1} (3x^2 - x^2)dydx = \int_{-1}^{1}\int_{-1}^{1} (2x^2)dydx$

$= \int_{-1}^{1} 2\left[\frac{x^3}{3} \right]_{-1}^{1} dy = \int_{-1}^{1} 2\left[\frac{1}{3} + \frac{1}{3} \right] dy = \int_{-1}^{1} \left[\frac{4}{3} \right] dy = \left[\frac{4}{3} \right] [y]_{-1}^{1} = \frac{8}{3} \qquad \rightarrow (1)$

CONT…

Consider

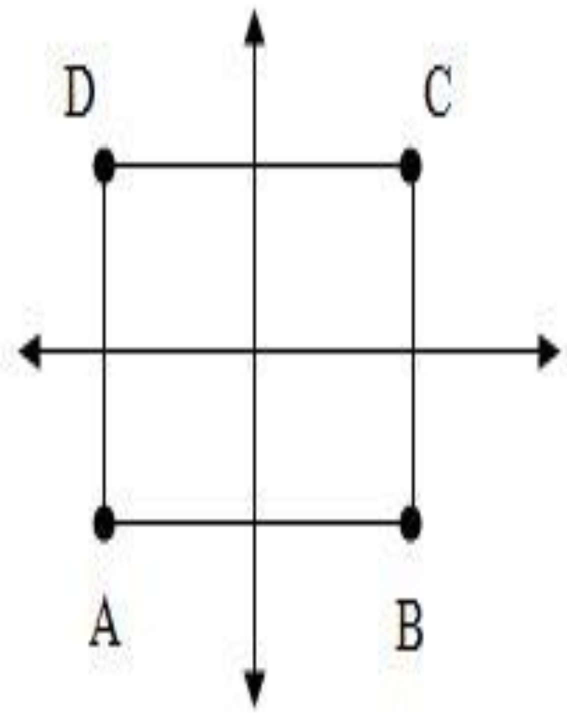
$$\int_C Pdx + Qdy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $y = -1$, $dy = 0$ and x varies from -1 to 1

$$\therefore \int_{AB} Pdx + Qdy = \int_{-1}^{1} x^2(1-1)dx = 0$$

Along BC, $x = 1$, $dx = 0$ and y varies from -1 to 1

$$\therefore \int_{BC} Pdx + Qdy = \int_{-1}^{1} (y^3 + 1)dy = \left[\frac{y^4}{4} + y\right]_{-1}^{1} = 2$$



Cont...

Along CD, $y = 1$, $dy = 0$ and x varies from 1 to -1

$$\therefore \int_{CD} Pdx + Qdy = \int_{1}^{-1} 2x^2 dx = \left[\frac{2x^3}{3}\right]_1^{-1} = -\frac{4}{3}$$

Along DA, $x = -1$, $dx = 0$ and y varies from 1 to -1

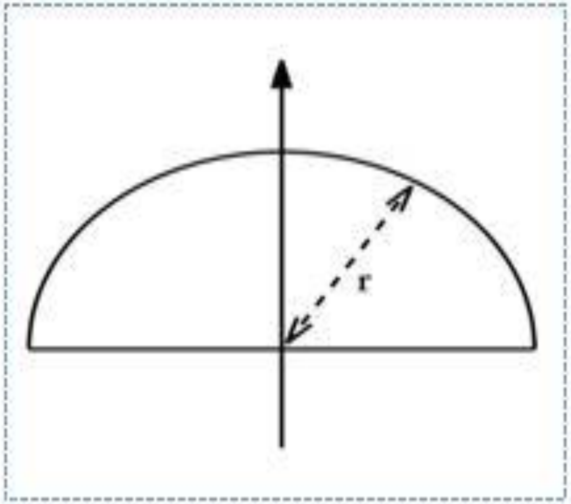
$$\therefore \int_{DA} Pdx + Qdy = \int_{1}^{-1} (y^3 - 1)dy = \left[\frac{y^4}{4} - y\right]_1^{-1} = \frac{1}{4} + 1 - \frac{1}{4} + 1 = 2$$

$$\int_C Pdx + Qdy = 0 + 2 - \frac{4}{3} + 2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad \rightarrow (2)$$

$\therefore$ (1) = (2) Hence the theorem is verified.

PROBLEM 9

Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$, where C is the boundary of the area bounded by the x axis and the upper half of the circle $x^2 + y^2 = a^2$



Solution

By Green's Theorem $\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \right) dxdy$

Given $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ $P = (2x^2 - y^2) \Rightarrow \frac{\partial P}{\partial y} = -2y$;

$Q = (x^2 + y^2) \Rightarrow \frac{\partial Q}{\partial x} = 2x \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \right) = (2x + 2y) = 2(x + y)$

$\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \right) dxdy = 2\iint_A (x + y)\, dxdy$,

where A is the region of upper half of the circle $x^2 + y^2 = a^2$

Cont...

Changing into polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$ and the limits are

$r : 0$ *to* a, $\theta = 0$ *to* π.

$$\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \right) dxdy = 2\int_0^{\pi} \int_0^{a} r\,(\cos\theta + \sin\theta)\, r\, dr.d\theta = 2\int_0^{\pi} \int_0^{a} r^2\, dr\, .(\cos\theta + \sin\theta)\, d\theta$$

$$= 2.\left[\frac{r^3}{3} \right]_0^a .\,(-\sin\theta + \cos\theta)_0^{\pi} = \frac{2a^3}{3}\left((0-1)-(0+1)\right)$$

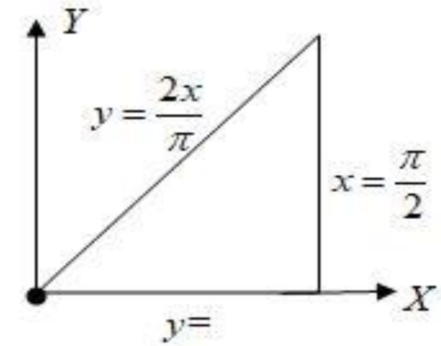
$$= \frac{4a^3}{3} \text{(numerically)}$$

Problem 10

Using Green's theorem , Evaluate $\int_C (y-\sin x)dx+\cos x\,dy$ where C is the plane triangle bounded by the lines y= 0, $x=\frac{\pi}{2}$ and $y=\left(\frac{2}{\pi}\right)x$

Solution: Green's theorem states that $\int_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

Given $\int_C (\mathrm{y}-\sin\mathrm{x})\mathrm{dx}+\cos\mathrm{x}\,\mathrm{dy}$



$$P=y-\sin x \Rightarrow \frac{\partial P}{\partial y}=1 \ ;\ Q=\cos x \quad \Rightarrow \frac{\partial Q}{\partial x}=-\sin x$$

$$\int_C (y-\sin x)dx+\cos x\,dy=\iint_R(-\sin x-1)\,dx\,dy$$

$$=\int_0^1\int_{\frac{\pi y}{2}}^{\frac{\pi}{2}}(-\sin x-1)\,dx\,dy$$

$$=\int_0^1[\cos x-x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}}dy=\int_0^1\left[\left(\cos\frac{\pi}{2}-\frac{\pi}{2}\right)-\left(\cos\frac{\pi y}{2}-\frac{\pi y}{2}\right)\right]dy$$

$$=\left[-\frac{\pi y}{2}-\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}+\frac{\pi y^2}{4}\right]_0^1 \quad =-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

PROBLEM 11

Verify Green's theorem in a plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the triangle formed by the lines $x = 0, y = 0$ and $x + y = 1$.

Solution:

Green's theorem states that

Given $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

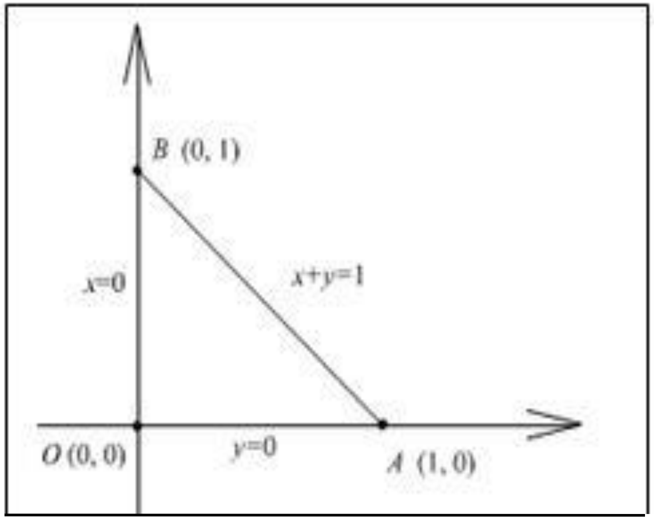
$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$P = 3x^2 - 8y^2$

$\frac{\partial P}{\partial y} = -16y$

$Q = 4y - 6xy$

$\frac{\partial Q}{\partial x} = -6y$



CONT…

Evaluation of RHS:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_R (-6y + 16y)\, dx\,dy$$

$$= \int_0^1 \int_0^{1-y} 10y\, dx\, dy = \int_0^1 10y\, [x]_0^{1-y}\, dy$$

$$= \int_0^1 10y(1 - y)\, dy$$

$$= 10 \int_0^1 (y - y^2)\, dy$$

Cont...

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{10}{6}$$

$$= \frac{5}{3}$$

Evaluation of LHS:

$$\int_C (Pdx + Qdy) = \int_{OA} (Pdx + Qdy) + \int_{AB} (Pdx + Qdy) + \int_{BO} (Pdx + Qdy)$$

CONT...

Along OA: $y=0 \Rightarrow dy=0$

$$\int_{OA} Pdx+Qdy=\int_{OA}\left(3x^2\right)dx$$

$$=\left[\frac{3x^3}{3}\right]_0^1=1-0=1$$

Along AB:

$$x+y=1 \Rightarrow y=1-x$$

$$\Rightarrow dy=-dx$$

$$\int_{AB} Pdx+Qdy=\int_{AB}\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy$$

$$=\int_{AB}[3x^2-8(1-x)^2]dx+[4(1-x)-6x(1-x)](-dx)$$

Cont...

$$= \int_1^0 \left(-11x^2 + 26x - 12\right) dx$$

$$= \left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0 = (0) - \left(\frac{-11}{3} + \frac{26}{2} - 12\right) = \frac{11}{3} - 1 = \frac{8}{3}$$

Along BO: $x = 0 \Rightarrow dx = 0$

$$\int_{BO} Pdx + Qdy = \int_{BO} 4y\,dy$$

$$= \left[\frac{4y^2}{2}\right]_1^0 = 2[0 - (1)]$$

$$= -2$$

$$\therefore \int_C Pdx + Qdy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

Hence Green's theorem is verified.

GAUSS DIVERGENCE THEOREM.

Statement: The surface integral of the normal component of a vector function $\vec{F}$ over a closed surface S enclosing the volume V is equal to the volume integral of the divergence of $\vec{F}$ taken throughout the volume V.

$$\iint_S \vec{F} \bullet \hat{n}\, dS = \iiint_V div\vec{F}\, dV = \iiint_V \nabla \bullet \vec{F}\, dV$$

, where $\vec{n}$ is the unit outward normal to the surface S.

PROBLEM 12

Use Gauss divergence theorem, prove that $\iint\limits_S \vec{r}.\hat{n}\,ds = 3V$, where V is the volume enclosed by the surface S.

Solution: By Gauss divergence theorem

$$\iint\limits_S \vec{r}\bullet\hat{n}\,ds = \iiint\limits_V \nabla\bullet\vec{r}\,dV = \iiint\limits_V \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\bullet(x\vec{i}+y\vec{j}+z\vec{k})\,dV = \iiint\limits_V \left(\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right)dV$$

$$= \iiint\limits_V 3\,dV$$

$$= 3V$$

Problem 13

Verify Gauss Divergence theorem for $\vec{F} = 4xz\,\vec{i} + y^2\,\vec{j} + yz\,\vec{k}$ over the cube bounded by x = 0, y = 0, z= 0, x =a, y =a and z =a.

Solution

By Gauss – Divergence theorem $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V div\vec{F}\cdot dV$

Evaluation of LHS:

$$\iint_S \vec{F}\cdot\vec{n}ds = \iint_{S_1} \vec{F}\cdot\overline{n}\,ds + \iint_{S_2} \vec{F}\cdot\overline{n}\,ds + \ldots + \iint_{S_6} \vec{F}\cdot\overline{n}\,ds$$

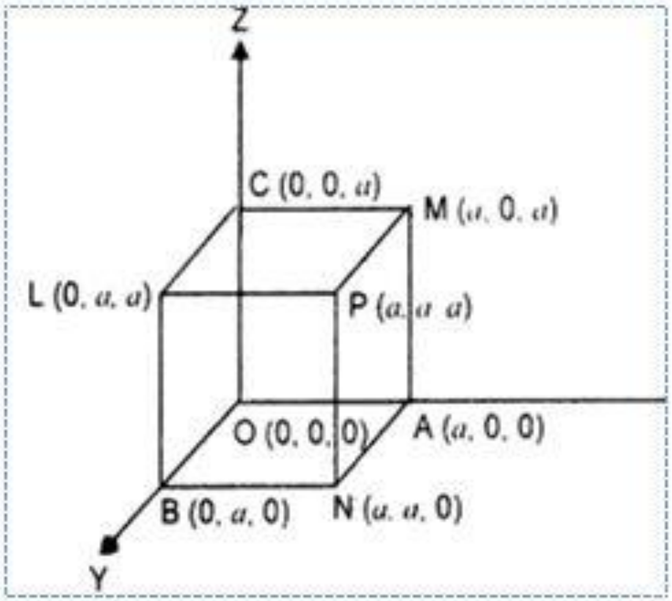
Over S_1: $\vec{n} = -\vec{i}, x = a$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,ds = \int_0^a\int_0^a (0)\,dy\,dz = 0$$

Over S_2: $\vec{n} = \vec{i}, x = a$

$$\iint_{S_2} \vec{F}\cdot\overline{n}\,ds = \int_0^a\int_0^a (4az)\,dy\,dz$$

$$= \int_0^a \left[4a\left(\frac{z^2}{2}\right)_0^a\right]dy = \int_0^a \left[4a\left(\frac{a^2}{2} - 0\right)\right]dy = \int_0^a 2a^3 dy = 2a^3[y]_0^a = 2a^4$$



Cont...

Over S_3: $\vec{n} = -\vec{j}, y = 0$

$$\iint_{S_3} \vec{F} \cdot \overline{n}\, ds = \int_0^a \int_0^a (0)\, dx\, dz = 0$$

Over S_4: $\vec{n} = \vec{j}, y = a$

$$\iint_{S_4} \vec{F} \cdot \overline{n}\, ds = \int_0^a \int_0^a (-a^2)\, dx\, dz = \int_0^a -a^2 [x]_0^a\, dz = \int_0^a -a^2 [a-0]\, dz = -a^2 (a)[z]_0^a = -a^4$$

Over S_5: $\vec{n} = \vec{k}, z = 0$

$$\iint_{S_5} \vec{F} \cdot \overline{n}\, ds = \int_0^a \int_0^a (0)\, dx\, dy = 0$$

Over S_6: $\vec{n} = \vec{k}, z = a$

$$\iint_{S_6} \vec{F} \cdot \overline{n}\, ds = \int_0^a \int_0^a \mathrm{a}(y)\, dx\, dy = \int_0^a a[x]_0^a y\, dy = \int_0^a a^2 y\, dy = \left[\frac{a^2 y^2}{2}\right]_0^a = \frac{a^4}{2}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = 0 + 2a^4 + 0 - a^4 + 0 + \frac{a^4}{2} = \frac{3a^4}{2}$$

CONT...

Evaluation of RHS:

$$\nabla.\vec{F} = 4z - y$$

$$\iiint_V \nabla.\vec{F}\, dV = \int_0^a\int_0^a\int_0^a (4z - y)\, dx\, dy\, dz$$

$$= \int_0^a\int_0^a \left[(4xz - xy)\right]_0^1 dy\, dz = \int_0^a\int_0^a \left[(4az - ay)\right] dy\, dz$$

$$= \int_0^a \left[\left(4ayz - \frac{ay^2}{2}\right)\right]_0^a dz = \int_0^a \left[\left(4a^2z - \frac{a^3}{2}\right)\right] dz = \left[\left(\frac{4a^2z^2}{2} - \frac{a^3z}{2}\right)\right]_0^a = \left(2a^4 - \frac{a^4}{2}\right) = \frac{3a^4}{2}$$

L.H.S = R.H.S

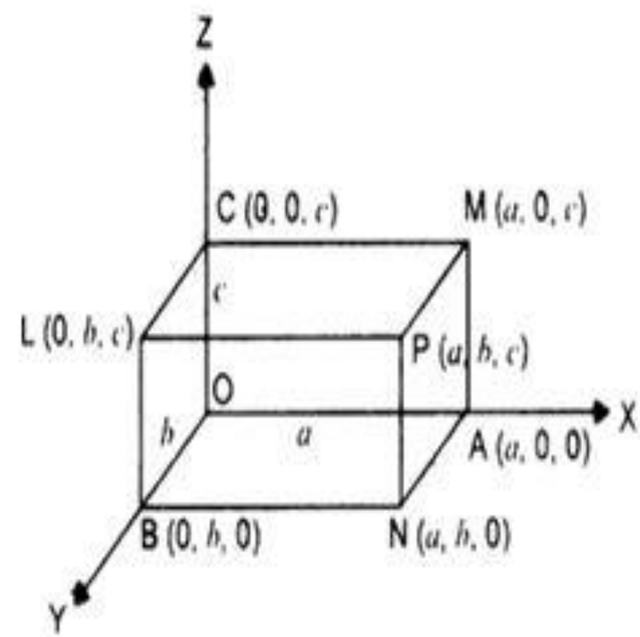
Hence, Gauss divergence theorem is verified.

PROBLEM 14

Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} - (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the cuboid bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$.

Solution:

By Gauss – Divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V div\vec{F} \cdot dV$



Evaluation of LHS:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \overline{n}\, ds + \iint_{S_2} \vec{F} \cdot \overline{n}\, ds + \ldots + \iint_{S_6} \vec{F} \cdot \overline{n}\, ds$$

Over S_1: $\vec{n} = -\vec{i}$, $x = 0$

$$\iint_{S_1} \vec{F} \cdot \overline{n}\, ds = \int_0^c \int_0^b (yz)\, dy\, dz$$

$$= \int_0^c \left[z \left(\frac{y^2}{2} \right)_0^b \right] dz = \frac{b^2}{2} \left(\frac{z^2}{2} \right)_0^c = \frac{b^2 c^2}{4}$$

CONT...

Over S_2: $\vec{n}=\vec{i}, x = a$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,ds = \int_0^c\int_0^b(-yz+a^2)\,dy\,dz = \int_0^b\left[-y\left(\frac{z^2}{2}\right)_0^c + a^2[z]_0^c\right]dy = -\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b + ca^2[y]_0^b = a^2bc-\frac{b^2c^2}{4}$$

Over S_3: $\vec{n}=-\vec{j}, y = 0$

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,ds = \int_0^c\int_0^a(xz)\,dx\,dz = \int_0^c\left(\frac{x^2}{2}z\right)_0^a dz = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4}$$

Over S_4: $\vec{n}=\vec{j}, y = b$

$$\iint_{S_4} \vec{F}\cdot\vec{n}\,ds = \int_0^c\int_0^a(-xz+b^2)\,dx\,dz = \int_0^c\left[-z\left(\frac{a^2}{2}\right)+b^2a\right]dz = ab^2c-\frac{a^2c^2}{4}$$

Over S_5: $\vec{n}=-\vec{k}, z = 0$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,ds = \int_0^b\int_0^a(xy)\,dx\,dy = \int_0^b\left[y\left(\frac{x^2}{2}\right)_0^a\right]dy = \frac{a^2b^2}{4}$$

Cont...

Over S_6: $\vec{n} = \vec{k}, z = c$

$$\iint_{S_6} \vec{F} \cdot \overline{n}\, ds = \int_0^b \int_0^a (-xy + c^2)\, dx\, dy = \int_0^b \left[-y\left(\frac{a^2}{2}\right) + c^2 a \right] dy = abc^2 - \frac{a^2 b^2}{4}$$

$$\therefore \iint_S \vec{F} \cdot \overline{n}\, ds = \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + a\, b^2 c - \frac{a^2 c^2}{4} + \frac{a^2 b^2}{4} + a\, bc^2 - \frac{a^2 b^2}{4}$$

$$= a^2 bc + ab^2 c + abc^2 = abc(a + b + c)$$

CONT...

Evaluation of RHS: $\nabla.\vec{F} = 2(x+y+z)$

$$\iiint_V \nabla \vec{F}\, dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + xy + xz\right]_0^a dy\,dz = 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\,dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + a\frac{y^2}{2} + ayz\right]_0^b dz = 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right] = a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

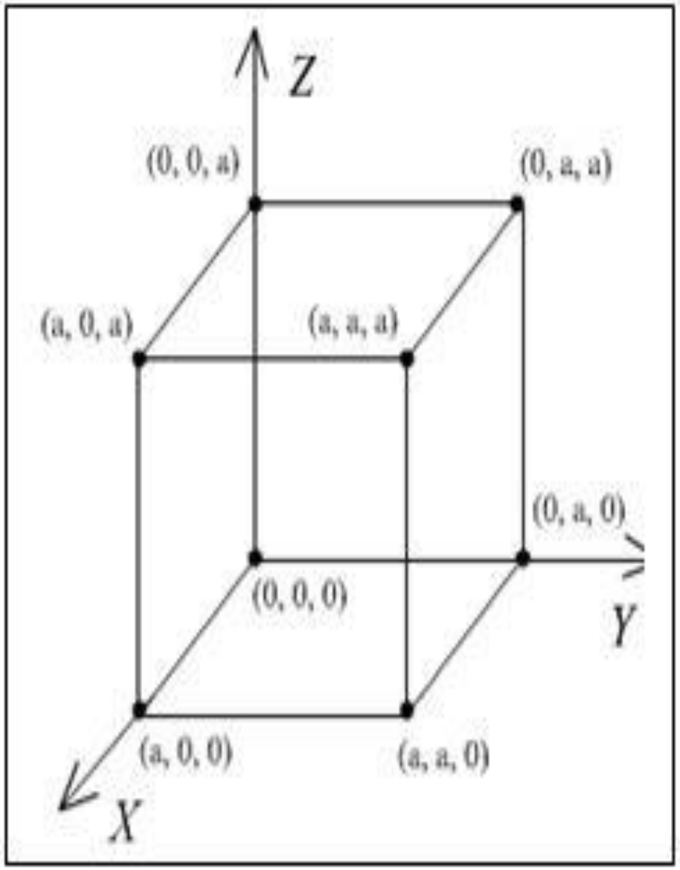
Hence, Gauss divergence theorem is verified.

PROBLEM 15

Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over the cube bounded by the planes x = 0, x = a, y = 0, y = a, z = 0 and z = a.
(A/M2018)

Solution:

By Gauss – Divergence theorem $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\,dV$



Evaluation of LHS:

$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}ds + \ldots + \iint_{S_6} \vec{F}\cdot\hat{n}ds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{i})\,dy\,dz = \int_0^a\int_0^a -x^3\,dy\,dz$$

$=0$

Cont…

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{i})\, dy\, dz = \int_0^a \int_0^a x^3\, dy\, dz$$

$$= \int_0^a \int_0^a a^3\, dy\, dz = a^3 \int_0^a [y]_0^a dz = a^3 \int_0^a a\, dz$$

=

$$= a^4 [z]_0^a = a^4 (a) = a^5$$

Cont...

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i}+y^3\vec{j}+z^3\vec{k}).(-\vec{j})\,dx\,dz = \int_0^a\int_0^a -y^3\,dx\,dz = 0$$

Over S_4: y = a, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i}+y^3\vec{j}+z^3\vec{k}).(\vec{j})\,dx\,dz = \int_0^a\int_0^a y^3\,dx\,dz$$

$$= \int_0^a\int_0^a a^3\,dx\,dz = a^3\int_0^a [x]_0^a\,dz = a^3\int_0^a [a-0]\,dz = a^4[z]_0^a = a^4(a) = a^5$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i}+y^3\vec{j}+z^3\vec{k}).(-\vec{k})\,dx\,dy = \int_0^a\int_0^a -z^3\,dx\,dy = 0$$

CONT....

Over S_6: z = a, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{k})\, dx\, dy = \int_0^a \int_0^a z^3\, dx\, dy$$

$$= a^3 \int_0^a [x]_0^a dy = a^3 \int_0^a a\ dy = a^4 [y]_0^a = a^4(a) = a^5$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = 0 + a^5 + 0 + a^5 + 0 + a^5 = 3a^5$$

Evaluation of RHS:

$$\nabla . \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \left(x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k} \right)$$

$$\nabla . \vec{F} = 3x^2 + 3y^2 + 3z^2$$

CONT…

$$\iiint_V \nabla \cdot \vec{F}\, dV = \int_0^a \int_0^a \int_0^a 3x^2 + 3y^2 + 3z^2 \, dx\, dy\, dz$$

$$= 3\int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2 \, dx\, dy\, dz$$

$$= 3\int_0^a \int_0^a \left[\frac{x^3}{3} + (y^2 + z^2)x \right]_0^a dy\, dz$$

$$= 3\int_0^a \int_0^a \left[\frac{a^3}{3} + (y^2 + z^2)\mathrm{a} \right] dy\, dz$$

$$= 3\int_0^a \left[\frac{a^3}{3} y + a\frac{y^3}{3} + az^2 y \right]_0^a dz$$

$$= 3\int_0^a \frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2 \, dz$$

$$= 3\left[\frac{a^4}{3} z + \frac{a^4}{3} z + a^2 \frac{z^3}{3} \right]_0^a$$

$$= 3\left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} \right] = \frac{9a^5}{3} = 3a^5 .$$

Stoke's theorem.

Statement: The surface integral of the normal component of the curl of a vector function F over an open surface S is equal to the line integral of the tangential component of F around the closed curve C bounding S

$$\int_C F \bullet dr = \iint_S curl\,\vec{F} \bullet \hat{n}\,ds = \iint_S \left(\nabla \times \vec{F}\right) \bullet \hat{n}\,ds$$

Problem 16

If S is any closed surface enclosing a volume V and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, prove that

$$\iint_S \vec{F}.\hat{n}\, ds = (a+b+c)V$$

Solution: $\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla \bullet \vec{F}\, dV = \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \bullet (ax\vec{i} + by\vec{j} + cz\vec{k})dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dv = (a+b+c)V$$

PROBLEM 17

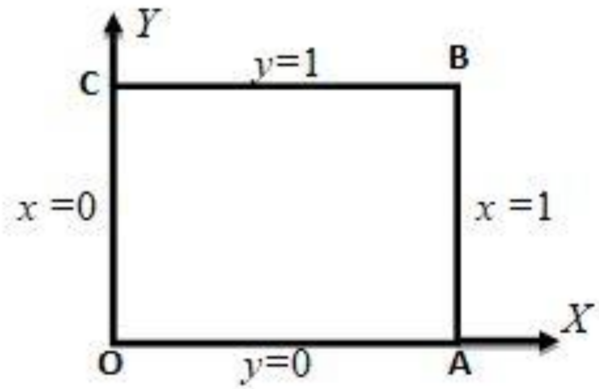
Verify Stoke's theorem for the vector field defined by $\vec{F} = (x^2 + y^2)\vec{i} + 2xy\vec{j}$ taken around the square bounded by the lines $x = 0$, $x = 1$, $y = 0$, $y = 1$.

Solution:

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\,\hat{n}\,ds$

Given $\vec{F} = (x^2 + y^2)\vec{i} + 2xy\vec{j}$

$\vec{F}.d\vec{r} = (x^2 + y^2)dx + 2xy\,dy$



Cont...

Evaluation of LHS:

$$\int_C \overline{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA : $y = 0 \Rightarrow dy = 0$, x varies from 0 to 1

$$\therefore \int_{OA} \overline{F}.d\vec{r} = \int_0^1 (x^2 + y^2)dx + 2xy\ dy = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$$

Along AB: $x = 1 \Rightarrow dx = 0$, y varies from 0 to 1

$$\int_{AB} \overline{F}.d\vec{r} = \int_0^1 (1 + y^2).0 - 2y\ dy = -2\left(\frac{y^2}{2}\right)_0^1 = -1$$

Along BC: $y = 1$, $dy = 0$, x varies from 1 to 0

$$\int_{BC} \overline{F}.d\vec{r} = \int_1^0 (x^2 + 1)dx - 0 = \left(\frac{x^3}{3} + x\right)_1^0 = -\left(\frac{1}{3} + 1\right) = -\frac{4}{3}$$

Along CO: $x = 0$, $dx = 0$, y varies from 1 to 0

$$\int_{CO} \overline{F}.d\vec{r} = \int_1^0 (0 + y^2)0 + 0 = 0$$

$$\therefore \int_C \overline{F}.d\vec{r} = \int_{OA} \overline{F}.d\vec{r} + \int_{AB} \overline{F}.d\vec{r} + \int_{BC} \overline{F}.d\vec{r} + \int_{CO} \overline{F}.d\vec{r} = \frac{1}{3} - 1 - \frac{4}{3} = -2.$$

CONT...

Evaluation of RHS:

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & 2xy & 0 \end{vmatrix} = \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the xy plane we can take $\hat{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\hat{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dxdy = -4\int_0^1\int_0^1 y\,dxdy = -4\left(\frac{y^2}{2}\right)_0^1 (x)_0^1 = -2.$$

$$\therefore \int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\,ds$$

Hence Stoke's theorem is verified.

PROBLEM 18

Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ where S is the open surface of the cube bounded by $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$ and $z = a$. above the XOY plane.

Solution:

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\,ds$

Evaluation of $L.H.S$:

$\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$

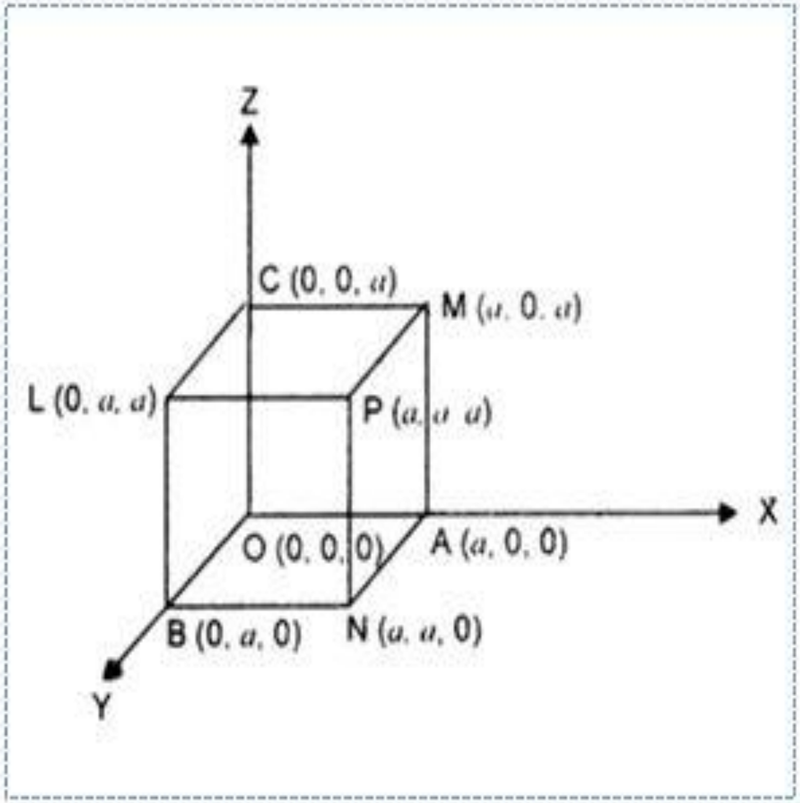
$\vec{F}.d\vec{r} = (y - z + 2)dx + (yz + 4)dy - xzdz$

As the region is in the xy plane we can take

$z = 0 \Rightarrow dz = 0.$

$\therefore \vec{F}.d\vec{r} = (y + 2)dx + 4dy$

$\int_C \vec{F}.\overline{dr} = \int_{OA} \vec{F}.\overline{dr} + \int_{AN} \vec{F}.\overline{dr} + \int_{NB} \vec{F}.\overline{dr} + \int_{BO} \vec{F}.\overline{dr}$



Cont...

Along OA: $y=0,\ dy=0,\ x: 0\ to\ a$

$$\int_{OA} \vec{F}.\overline{dr} = \int_0^a 2dx = 2[x]_0^a = 2a.$$

Along AN: $x=a,\ dx=0\ ; y=0\ to\ a$

$$\int_{AB} \vec{F}.\overline{dr} = \int_0^a (y+2)(0)+4dy = 4[y]_0^a = 4a.$$

Along NB: $y=a,\ \ dy=0, x: a\, to\, 0$

$$\int_{BD} \vec{F}.\overline{dr} = = \int_a^0 (a+2)\ dx = (a+2)[x]_a^0 = -(a^2+2a)$$

Along BO: $x=0,\ dx=0\ ; y=a\ to\ 0$

$$\int_{DO} \vec{F}.\overline{dr} = \int_a^0 (y+2)(0)+4dy = 4[y]_a^0 = -4a$$

$$\therefore \int_C \vec{F}.\overline{dr} = 2a+4a-a^2-2a-4a = -a^2.$$

CONT...

Evaluation of RHS:

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \vec{i}[0-y] - \vec{j}[-z+1] + \vec{k}[0-1] = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint_{S_1} curl\,\vec{F}.\vec{n}\,ds + \iint_{S_2} curl\,\vec{F}.\vec{n}\,ds + \ldots + \iint_{S_5} curl\,\vec{F}.\vec{n}\,ds$$

(Since S is open surface)

Over S_1: $\hat{n} = -\vec{i}$, $x = 0$

$$\iint_{S_1} \nabla \times \vec{F}.\hat{n}\,ds = \int_0^a\int_0^a \left[-y\vec{i}\right].(-\vec{i})\,dydz = \int_0^a\int_0^a [y].dy\,dz = \left[\frac{y^2}{2}\right]_0^a [z]_0^a = \frac{a^3}{2}$$

Over S_2: $\hat{n} = \vec{i}$, $x = a$

$$\iint_{S_2} \nabla \times \vec{F}.\hat{n}\,ds = \int_0^a\int_0^a \left[-y\vec{i}\right].(\vec{i})\,dydz = \int_0^a\int_0^a [-y].dy\,dz = -\left[\frac{y^2}{2}\right]_0^a [z]_0^a = -\frac{a^3}{2}$$

Cont...

Over S_3: $\hat{n} = -\vec{j}$, $y = 0$

$$\iint_{S_3} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^a\int_0^a \left[(z-1)\vec{j}\right].(-\vec{j})dydz = \int_0^a\int_0^a [-(z-1)].\mathrm{d}xdz = -\left[\frac{z^2}{2} - z\right]_0^a [x]_0^a = -\left[\frac{a^3}{2} - a^2\right]$$

Over S_4: $\hat{n} = \vec{j}$, $y = a$

$$\iint_{S_4} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^a\int_0^a \left[(z-1)\vec{j}\right].(\vec{j})dxdz = \int_0^a\int_0^a [(z-1)].\mathrm{d}xdz = \left[\frac{z^2}{2} - z\right]_0^a [x]_0^a = \left[\frac{a^3}{2} - a^2\right]$$

Over S_5 : $\hat{n} = \vec{k}$, $z = a$

$$\iint_{S_5} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^a\int_0^a \left[-\vec{k}\right].(\vec{k})dxdy = -\int_0^a\int_0^a dxdy = -[x]_0^a [y]_0^a = -a^2.$$

$\therefore$ L.HS = R.HS. Hence Stoke's theorem is verified.

Problem 19

Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\,\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelepiped formed by the planes x = 0, y = 0, z=0, x = 1, y = 2 and z = 3 above the XOY plane.

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\,ds$

Evaluation of $L.H.S$:

$$\int_C \vec{F}.\overline{dr} = \int_{OA} \vec{F}.\overline{dr} + \int_{AB} \vec{F}.\overline{dr} + \int_{BD} \vec{F}.\overline{dr} + \int_{DO} \vec{F}.\overline{dr}$$

Along OA : $y = 0,\ z = 0,\ dy = 0,\ dz = 0$

$$\int_{OA} \vec{F}.\overline{dr} = 0$$

CONT...

Along AB : $x = 1,\ z = 0,\ dx = 0,\ dz = 0$

$$\int_{AB} \vec{F}.\overrightarrow{dr} = \int_{AE} 0 = 0$$

Along BD : $y = 2,\ z = 0,\ dy = 0, dz = 0$

$$\int_{BD} \vec{F}.\overrightarrow{dr} = \int_{BD} (2x)dx = \int_{1}^{0} 2x\,dx = \left[\frac{2x^2}{2}\right]_1^0 = 0 - 1 = -1$$

Along DO: $x = 0,\ z = 0, dx = 0, dz = 0$

$$\int_{DO} \vec{F}.\overrightarrow{dr} = \int_{DO} 0 = 0$$

$$\therefore LHS = \int_{C} \vec{F}.\overrightarrow{dr} = 0 + 0 - 1 + 0 = -1$$

Cont...

Evaluation of RHS:

$$\iint_S \nabla\times\vec{F}.\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Given, $\vec{F} = (xy)\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = 2y\vec{i} + (-z)\vec{j} - x\vec{k}$$

Over S_1: $x = 0$, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \nabla\times\vec{F}.\hat{n}\,ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].(-\vec{i})dydz = \int_0^3\int_0^2 -2y\,dy\,dz = \int_0^3\int_0^2 -2y\,dy\,dz = \int_0^3\left[\frac{-2y^2}{2}\right]_0^2 dz = -4(z)_0^3 = -12$$

Over S_2: $x = 1$, $\hat{n} = \vec{i}$

Cont...

$$\iint_{S_2} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].(\vec{i})\, dydz = \int_0^3\int_0^2 2y\, dy\, dz = \int_0^3 \left[\frac{2y^2}{2}\right]_0^2 dz = 12$$

Over S3: $y = 0$, $\hat{n} = -\vec{j}$

$$\iint_{S_1} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^3\int_0^1 \left[-z\vec{j}\right]\left(-\vec{j}\,\right) dxdz = \int_0^3\int_0^1 (z)\, dx\, dz = \int_0^3 (xz)_0^1 = \int_0^3 (z)\, dz = \left(\frac{z^2}{2}\right)_0^3 = \frac{9}{2}$$

Over S4: $y = 1$, $\hat{n} = \vec{j}$

$$\iint_{S_4} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^3\int_0^1 -z\vec{j}.\vec{j}\, dxdz = \int_0^3\int_0^1 (-z)\, dx\, dz = \int_0^3 (-xz)_0^1\, dz \qquad = \left(\frac{-z^2}{2}\right)_0^3 = -\frac{9}{2}$$

Over S5: $z = 1$, $\hat{n} = \vec{k}$

$$\iint_{S_5} \nabla \times \vec{F}.\hat{n}\, ds = \int_0^2\int_0^1 \left(-x\vec{k}\right).\vec{k}dxdy = \int_0^2\int_0^1 (-x)dxdy = \int_0^2 \left(-\frac{x^2}{2}\right)_0^1 dy = \int_0^2 \left(\frac{-1}{2}\right) dy = \left(\frac{-1}{2}\right)(y)_0^2 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} \quad = -12 + 12 + \frac{9}{2} - \frac{9}{2} - 1 = -1$$

$\therefore$ L.HS = R.HS.

Hence Stoke's theorem is verified.

Problem 20

Verify Stoke's theorem for $\vec{F} = \left(x^2 - y^2\right)\vec{i} + 2xy\,\vec{j}$ in the rectangular region of the z=0 plane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b$.(N /D2019)

Given $\vec{F} = \left(x^2 - y^2\right)\vec{i} + 2xy\,\vec{j}$

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}\,ds$

Evaluation of LHS:

$$\int_C \vec{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA : $y = 0 \Rightarrow dy = 0$, x varies from 0 to a

$$\therefore \int_{OA} \vec{F}.dr = \int_0^a \left(x^2\right)dx$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Cont...

Along AB: $x = a \Rightarrow dx = 0$, y varies from 0 to b

$$\int_{AB} \vec{F}.dr = \int_0^b 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b = ab^2$$

Along BC: $y = b$, $dy = 0$, x varies from a to 0

$$\int_{BC} \vec{F}.d\vec{r} = \int_a^0 \left(x^2 - b^2\right) dx$$

$$= \left(\frac{x^3}{3} - b^2 x\right)_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along CO: $x = 0$, $dx = 0$, y varies from b to 0

CONT…

$$\int_{CO} \vec{F}.d\vec{r} = \int_b^0 (0+y^2)0+0 = 0$$

$$\therefore \int_C \vec{F}.d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2$$

Evaluation of RHS:

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[2y+2y] = 4y\vec{k}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint 4y\vec{k}.\vec{k}\,dxdy$$

$$= 4\int_0^b\int_0^a y\ dxdy$$

$$= 4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= 2ab^2$$

$$\therefore \int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}ds$$

Hence Stoke's theorem is verified.

Thank you