UNIT - III

INTEGRAL CALCULUS

Definition:

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximately rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Definite Integral:

The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Note:

The definite integral $\int_a^b f(x)\,dx$ is a number, it does not depend on x.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr.$$

Riemann sum:

The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called a Riemann sum, $\Delta x = \frac{b-a}{n}$; $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals.

Theorem: 1 If f is continuous on $[a,b]$ or if f has only a finite number of jump discontinuous, then f is integrable on $[a,b]$.

i.e., $\int_a^b f(x)\,dx$ exists.

Theorem: 2

If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

NOTE:

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
4. $\sum_{i=1}^{n} c = nc$
5. $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$
6. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

problems based on definite integrals:

1. Evaluate $\int_0^3 (x^2 - 2x)\,dx$ by using Riemann sum by taking right end points as the sample points.

[U.Q: 2016]

Solution: Given: $\int_0^3 (x^2 - 2x)\,dx$.

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{6}{n},\ x_3 = \frac{9}{n} \ldots x_i = \frac{3i}{n}$

Since we are using right end points.

$$\int_0^3 (x^2 - 2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9}{n^2} i^2 - \frac{6}{n} i\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1+\frac{1}{n}\right]\left[2+\frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1+\frac{1}{n}\right]$$

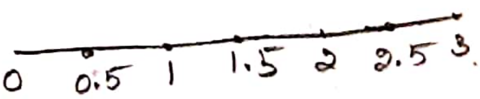
$$= \left(\frac{27}{6}\right)(1)(2) - 9 = 9 - 9 = 0$$

$$\therefore \int_0^3 (x^2 - 2x)\,dx = 0.$$

2. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right end points and $a = 0$, $b = 3$ and $n = 6$.

Solution: Given: $f(x) = x^3 - 6x$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$



$$\Delta x = \frac{1}{2}$$

The right end points are $0.5, 1, 1.5, 2, 2.5$ and 3.

$\therefore$ The Riemann sum is,

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= \sum_{i=1}^{6} f(x_i)\left(\frac{1}{2}\right) = \frac{1}{2} \sum_{i=1}^{6} f(x_i)$$

$$= \frac{1}{2}\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\right]$$

$$= \frac{1}{2}\left[-2.875 - 5 - 5.625 - 4 + 0.625 + 9\right]$$

$$R_6 = -3.9375$$

3. Set up an expression for $\int_1^3 e^x\,dx$ as a Limit of sums.

Solution: Given: $\int_1^3 e^x\,dx$

Here $f(x) = e^x$, $a = 1$, $b = 3$. $x_0 = 1$

$x_3 = 6/n$

and $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$ $\quad x_1 = 2/n$, $x_2 = 4/n$, $\vdots$ $x_i = \frac{2i}{n}$

So, $x_0 = 1$, $x_1 = 1 + \frac{2}{n}$, $x_2 = 1 + \frac{4}{n}$, $x_3 = 1 + \frac{6}{n}$ and

$x_i = 1 + \frac{2i}{n}$.

Now, $\int_1^3 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ $= \lim_{n\to\infty} \sum_{i=1}^{n} 2/n\, f\left(\frac{2i}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n}$ $= \lim_{n\to\infty} 2/n \sum_{i=1}^{n} e^{2i/n}$.

$$\int_1^3 e^x\,dx = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{1+\frac{2i}{n}}.$$

H.W.

1. Evaluate $\int_0^3 (x^3 - 6x)\,dx$ by using Riemann Sum by taking right end points as the sample points.

Sol: $\Delta x = \frac{3}{n}$, $x_0 = 0$, $x_1 = \frac{3}{n}$, $x_2 = \frac{6}{n}$, $x_3 = \frac{9}{n}$, ...

$x_i = \frac{3i}{n}$ $\quad$ Ans: $\int_0^3 (x^3 - 6x)\,dx = -6.75 = -\frac{27}{4}$.

Midpoint Rule:

③

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$= \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

where, $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$= \text{Midpoint of } [x_{i-1}, x_i].$

1. Use the mid point Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

③

Solution: Given: $\int_1^2 \frac{1}{x}\,dx$

The end points of the five sub-intervals are,

1 1.2 1.4 1.6 1.8 2

End points	1	1.2	1.4	1.6	
Mid points	1.1	1.3	1.5	1.7	1.9
$f(x) = 1/x$	$1/1.1$	$1/1.3$	$1/1.5$	$1/1.7$	$1/1.9$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

$$\int_1^2 \frac{1}{x}\,dx = \Delta x\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$= \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]$$

$$\approx 0.691908$$

Since $f(x) = 1/x > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation given by the Midpoints Rule is the sum of the areas of the rectangles.

The Fundemental Theorem of calculus. Part - I.

If f is continuous on $[a,b]$, then the function g is defined by,

$$g(x) = \int_a^x f(t)\, dt \quad , a \le x \le b.$$

is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$. $\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\int_a^x f(t)dt\right]$

$g'(x) = f(x)$

Find the derivative of the following functions:

1. $g(x) = \int_1^x \frac{1}{t^3+1}\, dt$

solution: Given: $g(x) = \int_1^x \frac{1}{t^3+1}\, dt.$

$\Rightarrow \quad g'(x) = \frac{1}{x^3+1}$ [by FTC I]
[$\because f(t) = t^3+1$ is continous].

2. $y(x) = \int_x^5 3t \sin t\, dt$

Solution: Given: $y(x) = \int_x^5 3t \sin t\, dt = -\int_5^x 3t \sin t\, dt$

$\therefore y'(x) = -3x \sin x$ [$\because f(t) = 3t \sin t$ is con]

3. $y(x) = \int_1^{x^2} \cos t\, dt$

Solution: Given: $y(x) = \int_1^{x^2} \cos t\, dt$

Put $u = x^2$, $du = 2x\,dx \Rightarrow \frac{du}{dx} = 2x.$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y'(x) = \left[\frac{d}{du}\left[\int_1^u \cos t\, dt\right]\right](2x)$ [by FTC I]

$= \cos u \; 2x \quad = 2x \cos x^2$

$y'(x) = 2x \cos x^2$

4. $h(x) = \int_1^{e^x} \log t \, dt.$ H.W

solution: **Given:** $h(x) = \int_1^{e^x} \log t \, dt$

put $u = e^x \Rightarrow du = e^x dx$

$$\Rightarrow \frac{du}{dx} = e^x$$

$$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left(\int_1^u \log t \, dt\right) e^x$$

$$= \log u \, (e^x) = \log(e^x) \cdot e^x = x e^x.$$

$$h'(x) = xe^x.$$

5. Find the derivative of the function $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2} du.$ by using the fundamental theorem of Calculus.

Solution: **Given:** $y(x) = \int_{1-3x}^{1} \frac{u^3}{1+u^2} du.$

put $t = 1-3x$, $dt = -3dx \Rightarrow \frac{dt}{dx} = -3.$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\left[\int_t^1 \frac{u^3}{1+u^2} du\right](-3)$$

$$= \frac{d}{dt}\left[-\int_1^t \frac{u^3}{1+u^2} du\right](-3) = \left(\frac{-t^3}{(1+t^2)}\right)(-3)$$

$$= \frac{3(1-3x)^3}{1+(1-3x)^2}.$$

$$y'(x) = \frac{3(1-3x)^3}{(1+(1-3x)^2)}.$$

H.W.

6. $y(x) = \int_{\sqrt{x}}^{0} \sin(t^2) \, dt.$ Ans: $y'(x) = -\frac{\sin x}{2\sqrt{x}}$

7. $y(x) = \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} dt$, $|x| < \frac{\pi}{2}$. Ans: $y'(x) = 1.$

The fundamental Theorem of Calculus part - II

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Where F is any anti-derivatives of f, that is a function such that $F' = f$.

1. Find the derivative of the following functions.

H.W $\int_1^3 e^x dx$

Solution: Given: $\int_1^3 e^x dx.$

Here, $f(x) = e^x$; Antiderivative $F(x) = e^x$.

$$\therefore \int_1^3 e^x dx = F(3) - F(1) = e^3 - e^1 \quad [\text{by FTC2}]$$

$$\int_1^3 e^x dx = e^3 - e$$

2. Evaluate $\int_3^6 \frac{1}{x} dx.$

Solution: Given: $\int_3^6 \frac{1}{x} dx$ $\quad [\because \log a - \log b = \log(a/b)]$

Here, $f(x) = \frac{1}{x}$, A.d $F(x) = \log x$

$$\int_3^6 \frac{1}{x} dx = F(6) - F(3) = \log 6 - \log 3 = \log\left(6/3\right) = \log 2.$$

$$\int_3^6 \frac{1}{x} dx = \log 2$$

3. prove that the following integral by interpreting each in terms of areas $\int_a^b x\, dx = \frac{b^2 - a^2}{2}$. [U.Q : 2016]

Solution: Given: $\int_a^b x\, dx$

Here, $f(x) = x$, A.d $F(x) = \frac{x^2}{2}$.

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$\therefore$ Area $= F(b) - F(a) = \frac{b^2}{2} - \frac{a^2}{2}$ [by FTC2] ⑤

Area $= \frac{b^2 - a^2}{2}$

4. Evaluate the following integrals $\int_{-1}^{2} (x^3 - 2x)\, dx$.

②! H.W

Solution: Given: $\int_{-1}^{2} (x^3 - 2x)\, dx$

Here, $f(x) = x^3 - 2x$, A.d $F(x) = \frac{x^4}{4} - \frac{2x^2}{2}$
$= \frac{x^4}{4} - x^2$.

$$A = \int_{-1}^{2} (x^3 - 2x)\, dx = F(2) - F(-1)$$

$$= \left[\frac{2^4}{4} - 2^2\right] - \left[\frac{(-1)^4}{4} - (-1)^2\right] = [4-4] - \left[\frac{1}{4} - 1\right]$$

$$A = \frac{3}{4}.$$

5. What is wrong with the equation? $\int_{\pi/3}^{\pi} \sec\theta \tan\theta\, d\theta$

$f(x) = \sec\theta \tan\theta$ $F(x) = \sec\theta$. $\int \sec x \tan x\, dx = \sec x$.

Solution: $\int_{\pi/3}^{\pi} \sec\theta \tan\theta\, d\theta = [\sec\theta]_{\pi/3}^{\pi} = -3$. $\sec\theta = \frac{1}{\cos\theta}$

$= F(\pi) - F(\pi/3)$ $\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2$ $= -1-2 = -3$. $\sec\pi = \frac{1}{\cos\pi} = -1$

$= \sec\pi - \sec\pi/3 = -1-2 = -3$

The function $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. $[\because \tan\frac{\pi}{2} = \infty]$.

6. What is wrong with the equation $\int_{-1}^{2} \frac{4}{x^3}\, dx$. [U.Q: Jan: 2016]

⑤ Solution: Given: $\int_{-1}^{2} \frac{4}{x^3}\, dx$.

$$\int_{-1}^{2} \frac{4}{x^3}\, dx = \int_{-1}^{2} 4\left[x^{-3}\right] dx = 4\left[\frac{x^{-2}}{-2}\right]_{-1}^{2} = -2\left[(2)^{-2} - (-1)^{-2}\right]$$

Here, $f(x) = 4x^{-3}$.

$$= -2\left[\frac{1}{2^2} - \frac{1}{1^2}\right] = -2\left[\frac{1}{4} - 1\right]$$

$$= -2\left(-3/4\right) = 3/2.$$

The function $f(x) = 1/x^3$ is not continuous on $[-1, 2]$.

The function $f(x)$ has an infinite discontinuity at $x = 0$.

$\therefore \int_{-1}^{2} 4/x^3\, dx$ does not exist.

H.W.

1. Find the area under the parabola $y = x^2$ from 0 to 1.

2. $\int_0^{\pi/4} \sec^2 t \, dt$.

3. $\int_1^9 \frac{x-1}{\sqrt{x}} \, dx$.

5.1. (b) Indefinite Integrals:

1. Evaluate the indefinite integrals $\int x^{-4} dx$.

Solution: Given: $\int x^{-4} dx$

$$= \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C.$$

2. $\int \frac{x^3+2x+1}{x^4} dx$

Solution:
Given: $\int \frac{x^3+2x+1}{x^4} dx = \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4} \right) dx$

$$= \int \left(\frac{1}{x} + 2x^{-3} + x^{-4} \right) dx.$$

$$= \log x + 2 \frac{x^{-2}}{(-2)} + \frac{x^{(-3)}}{-3} + C.$$

$$= \log x - \frac{1}{x^2} - \frac{1}{3x^3} + C.$$

H.W

3. $\int \left(x + \frac{1}{x}\right)^2 dx$ ∴ Ans: $\frac{1}{3}x^3 - \frac{1}{x} + 2x + C.$

4. $\int (e^{2x} + 3x - 7) \, dx$ Ans: $\frac{3x^2}{2} + \frac{e^{2x}}{2} - 7x + C.$

5. $\int (u+4)(2u+1) \, dx$ Ans: $\frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C.$

6. Find the general indefinite integral $\int (10x^4 - 2\sec^2 x)\,dx$. 6

Solution: Given: $\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$

$= 10\left[x^5/5\right] - 2\tan x + C$

$= 2x^5 - 2\tan x + C.$

7. $\int \frac{\sin^2 x}{1+\cos x}\,dx.$

Q

Solution: Given: $\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx$ $\left[\sin^2 x = 1-\cos^2 x\right]$

$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}\,dx$

$= \int (1-\cos x)\,dx$

$= x - \sin x + C.$

8. $\int (\tan x - 2\cot x)^2\,dx$

Solution: Given: $\int (\tan x - 2\cot x)^2\,dx$

$= \int \left[\tan^2 x + 4\cot^2 x - 4\tan x \cot x\right] dx$

$= \int \left[\sec^2 x - 1 + 4(\operatorname{cosec}^2 x - 1) - 4\tan x \frac{1}{\tan x}\right] dx$

$= \int \left[\sec^2 x - 1 + 4\operatorname{cosec}^2 x - 4 - 4\right] dx.$

$= \int \left[\sec^2 x + 4\operatorname{cosec}^2 x - 9\right] dx$

$= \tan x - 4\cot x - 9x + C.$

9. $\int \frac{1}{1+\sin x}\,dx$ Ans: $\tan x - \sec x + C.$

10. $\int \frac{\cos^2 x}{1-\sin x}\,dx$ Ans: $x - \cos x + C.$

Properties of definite integrals:

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)\,dx.$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)dx.$

6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx.$

7. i) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, iff $f(2a-x) = f(x)$

   ii) $\int_0^{2a} f(x)\,dx = 0$, iff $f(2a-x) = -f(x).$

8. If $f(x) = f(a+x)$ then $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx.$

9. (i) if $f(x)$ is an even function of x, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)dx.$

   (ii) If $f(x)$ is an odd function of x, then $\int_{-a}^{a} f(x)\,dx = 0.$

10. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

11. Max-Min inequality:

If f has Maximum value Max f and Minimum value min f on $[a, b]$, then,

$$\text{Min } f\,(b-a) \le \int_a^b f(x)\,dx \le \text{Max. } f\,(b-a).$$

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12. Domination:

$$f(x) \geq g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

$$f(x) \geq 0 \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0.$$

Problems based on properties of definite integrals:

1. $$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= 4\int_0^1 dx + 3\int_0^1 x^2\,dx = 4\left[x\right]_0^1 + 3\left[\frac{x^3}{3}\right]_0^1$$

$$= 4(1-0) + 3(1/3 - 0)$$

$$= 5$$

2. If $\int_0^{10} f(x)\,dx = 17$, $\int_0^{8} f(x)\,dx = 12$, then find $\int_8^{10} f(x)\,dx$.

Q

Solution: Given: $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$.

By property 5, $$\int_0^{10} f(x)\,dx = \int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx.$$

$$17 = 12 + \int_8^{10} f(x)\,dx.$$

$$\therefore \int_8^{10} f(x)\,dx = 17 - 12 = 5.$$

3. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$ (or) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\cot x}\,dx.$

Q

Solution: Given: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \rightarrow (1)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\therefore\ I = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx \rightarrow ②$$

$$①+② \Rightarrow\ 2I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin x}{\sin x+\cos x} + \frac{\cos x}{\cos x+\sin x}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx = \int_0^{\pi/2} 1\,dx = \left[x\right]_0^{\pi/2}$$

$$= \left[\pi/2 - 0\right] = \pi/2$$

$$\therefore\ I = \tfrac{1}{2}\cdot\pi/2 = \pi/4.$$

4. Evaluate $\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\,dx$ (or) $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx.$
Ⓠ

Ans: $I = \pi/4.$

5. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ (or) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx.$
②

Solution: Given: $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx. \rightarrow ①$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx. \rightarrow ②$$

$$①+② \Rightarrow\ 2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx.$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} \right] dx.$$ (8)

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} \right] dx$$

$$= \int_0^{\pi/2} 1 \, dx = [x]_0^{\pi/2} = [\pi/2 - 0] = \pi/2$$

$$I = \tfrac{1}{2} \cdot \pi/2 = \pi/4$$

$$I = \pi/4.$$

H.W

6. **Evaluate** $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx.$

Ans: $I = \pi/4.$

7. **Evaluate** $\int_0^1 \frac{\log(1+x)}{1+x^2} dx.$

Solution: **Given:** $\int_0^1 \frac{\log(1+x)}{1+x^2} dx.$

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx.$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

When $x=0$, $\tan\theta = 0 \Rightarrow \theta = 0$ and when $x=1$, $\tan\theta = 1$
$\Rightarrow \theta = \pi/4.$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(1+\tan^2\theta)} \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta \, d\theta. \quad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$= \int_0^{\pi/4} \log(1+\tan\theta) \, d\theta.$$

Let $I = \int_0^{\pi/4} \log(1+\tan x)\, dx \rightarrow (1)$

$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\pi/4 - x\right)\right] dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan x}{1+\tan x}\right] dx \quad \left[\because \tan(45^\circ - \theta) = \frac{1-\tan\theta}{1+\tan\theta}\right]$

$= \int_0^{\pi/4} \log\left[\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right] dx = \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right] dx$

$I = \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x)\right] dx \rightarrow (2) \quad \left[\because \log(a/b) = \log a - \log b\right]$

$(1)+(2) \Rightarrow \quad 2I = \int_0^{\pi/4} \log(1+\tan x)\,dx + \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x)\right] dx$

$= \int_0^{\pi/4} \log 2 \; dx = \log 2 \int_0^{\pi/4} 1 \cdot dx = \log 2 \left[x\right]_0^{\pi/4}$

$2I = \log 2\left[\pi/4 - 0\right] = \frac{\pi}{4} \log 2.$

$\therefore I = \frac{\pi}{8} \log 2.$

8. Evaluate $\int_0^{\pi/2} \log(\tan x)\, dx.$ Ans: $I = 0.$

9. Evaluate $\int_0^{\pi} x \log(\sin x)\, dx.$

(Q)

Solution: Given: $\int_0^{\pi} x \log(\sin x)\, dx.$

Let $I = \int_0^{\pi} x \log(\sin x)\, dx \rightarrow (1)$

$= \int_0^{\pi} (\pi - x) \log\left[\sin(\pi - x)\right] dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\therefore I = \int_0^{\pi} (\pi - x) \log(\sin x)\, dx. \rightarrow (2)$

9

① + ② $\Rightarrow$ $2I = \int_0^{\pi} x \log(\sin x)\, dx + \int_0^{\pi} (\pi - x) \log(\sin x)\, dx$

$$= \int_0^{\pi} [x \log(\sin x) + (\pi - x) \log(\sin x)]\, dx$$

$$= \int_0^{\pi} [x \log(\sin x) + \pi \log(\sin x) - x \log(\sin x)]\, dx.$$

$$= \int_0^{\pi} \pi \log(\sin x)\, dx.$$

$\therefore\ 2I = 2\pi \int_0^{\pi/2} \log(\sin x)\, dx$ $\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \text{ if } f(2a - x) = f(x)\right]$

$\Rightarrow\ I = \pi \left[-\frac{\pi}{2} \log 2\right]$ $\left[\because \int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2\right]$

$= \frac{\pi^2}{2} \log (2)^{-1}$ $[\because m \log n = \log n^m]$

$I = \frac{\pi^2}{2} \log (1/2)$ $[\log a^{-1} = \log 1/a]$.

10. $\int_0^{\pi} \sin^2 x \cos^3 x\, dx$ Ans: 0.

Solution: Given: $\int_0^{\pi} \sin^2 x \cos^3 x\, dx.$

$f(x) = \sin^2 x \cos^3 x\, dx.$

$f(-x) = \sin^2(-x)\cos^3(-x)\, dx.$

$= -\sin^2 x \cos^3 x\, dx$

$= -f(x).$

Let $I = \int_0^{\pi} \sin^2 x \cos^3 x\, dx.$

Let $f(x) = \sin^2 x \cos^3 x$

$\Rightarrow f(\pi - x) = \sin^2(\pi - x)\cos^3(\pi - x)$

$= \sin^2 x (-\cos x)^3$

$= -\sin^2 x \cos^3 x$

$= -f(x)$

$\therefore \int_0^{\pi} \sin^2 x \cos^3 x\, dx = 0$ $\left[\because \int_0^{2a} f(x)\, dx = 0, \text{ if } f(2a - x) = -f(x) \text{ Here } 2a = \pi\right].$

Substitution Rule:

$$\int [f(x)]^n f'(x)\, dx \quad \text{or} \quad \int \phi[f(x)] f'(x)\, dx.$$

substitute $u = f(x)$ and then proceed.

Algebraic function:

1\. Evaluate $\int (ax+b)^n dx.$

Solution: Given: $\int (ax+b)^n dx.$

Let $I = \int (ax+b)^n dx$

put $u = ax+b \Rightarrow du = a\,dx \Rightarrow \frac{du}{a} = dx.$

$$I = \int u^n \frac{du}{a} = \frac{1}{a}\int u^n du = \frac{1}{a}\frac{u^{n+1}}{n+1} + C$$

$$I = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C.$$

2\. Evaluate $\int \frac{x^2}{\sqrt{x+5}} dx.$

Solution: Given: $\int \frac{x^2}{\sqrt{x+5}} dx.$

Let $I = \int \frac{x^2}{\sqrt{x+5}} dx$

put $u = \sqrt{x+5}$, $du = \frac{1}{2\sqrt{x+5}} dx$

$$\Rightarrow 2\,du = \frac{1}{\sqrt{x+5}} dx$$

$$\Rightarrow dx = 2u\,du.$$

$$u^2 = x+5 \Rightarrow x = u^2-5 \Rightarrow x^2 = (u^2-5)^2 = u^4 - 10u^2 + 25$$

$$I = \int (u^4 - 10u^2 + 25)\, 2\,du = 2\int (u^4 - 10u^2 + 25)\,du.$$

$$= 2\left[\frac{u^5}{5} - 10\frac{u^3}{3} + 25u\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + C.$$

3. Evaluate $\int \sqrt{2x+1}\, dx$.

Solution: **Given:** $\int \sqrt{2x+1}\, dx$.

Let $I = \int \sqrt{2x+1}\, dx$.

put $u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx$

$\therefore dx = \frac{du}{2}$

$$I = \int \sqrt{u}\, du/_2$$

$$= \tfrac{1}{2} \int u^{1/2}\, du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$I = \frac{1}{3}(2x+1)^{3/2} + C.$$

4. H.W $I = \int \sqrt{ax+b}\, dx$.

5. Evaluate: $\int \frac{x}{\sqrt{1-4x^2}}\, dx$

Solution: **Given:** $\int \frac{x}{\sqrt{1-4x^2}}\, dx$

3

Let $I = \int \frac{x}{\sqrt{1-4x^2}}\, dx$

put $u = 1-4x^2$

$\frac{du}{dx} = -8x \Rightarrow du = -8x\, dx$

$\therefore dx = \frac{du}{-8x}$.

$$I = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-8x}$$

$$= -\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\int u^{-1/2} du = -\frac{1}{8}\left[\frac{u^{-1/2+1}}{-1/2+1}\right] + C$$

$$= -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$

$$I = -\frac{1}{4}(1-4x^2)^{1/2} + C.$$

$u = \sqrt{1-4x^2}$

$\frac{du}{dx} = \frac{1}{2}(1-4x^2)^{-1/2} \cdot (-8x)$

$\frac{du}{dx} = \frac{-4x}{\sqrt{1-4x^2}}$

$du = -4x\,dx$

$\frac{u}{\sqrt{1-4x^2}}\,du$

$u\,du$

-4

$x\,dx$

Logarithmic functions:

1. Evaluate $\int \frac{(\log x)^2}{x} dx$.

Q, **Solution:** Given: $\int \frac{(\log x)^2}{x} dx$

Let $I = \int \frac{(\log x)^2}{x} dx$.

put $u = \log x$; $du = \frac{1}{x} dx$

$\therefore I = \int u^2 du = \frac{u^3}{3} + C$.

$I = \frac{(\log x)^3}{3} + C$.

2. Evaluate $\int_1^e \frac{\log x}{x} dx$

Q

Solution: Given: $\int_1^e \frac{\log x}{x} dx$

Let $I = \int_1^e \frac{\log x}{x} dx$

Put $u = \log x$ $du = \frac{1}{x} dx$

$x \to 1 \Rightarrow u \to 0$; $x \to e \Rightarrow u \to 1$

$\therefore I = \int_0^1 u \, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

$I = 1/2$.

3. Evaluate $\int \frac{\sec^2(\log x)}{x} dx$.

solution: Given: $\int \frac{\sec^2(\log x)}{x} dx$

Let $I = \int \frac{\sec^2(\log x)}{x} dx$

put $u = \log x$, $du = 1/x \, dx$

$I = \int \sec^2 u \, du = \tan u + C = \tan(\log x) + C$.

Exponential functions:

1. Evaluate $\int e^{x^3} x^2\,dx$.

Solution: Given: $\int e^{x^3} x^2\,dx$.

Let $I = \int e^{x^3} x^2\,dx$

put $u = e^{x^3} \Rightarrow du = e^{x^3} \cdot 3x^2\,dx$

$\Rightarrow \frac{du}{3} = e^{x^3} \cdot x^2\,dx$

$I = \int \frac{du}{3} = \frac{1}{3}\int du = \frac{1}{3}u + C$

$I = \frac{1}{3} e^{x^3} + C$

2. Evaluate $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

Solution: Given: $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

put $u = e^{1/x} \Rightarrow du = e^{1/x}\left(-1/x^2\right) dx$

$\Rightarrow -du = \frac{e^{1/x}}{x^2}\,dx$

$x \to 1 \Rightarrow u \to e, \quad x \to 2 \Rightarrow u \to e^{1/2}$

$\therefore I = \int_e^{e^{1/2}} (-du) = -(u)_e^{e^{1/2}} = -\left[e^{1/2} - e\right]$

$I = e - \sqrt{e}$.

3. $\int e^{\cos x} \sin x\,dx$

(a)

Solution: Given: $\int e^{\cos x} \sin x\,dx$.

Let $I = \int e^{\cos x} \sin x\,dx$

put $u = e^{\cos x}$ $du = e^{\cos x}(-\sin x)\,dx$

$I = \int (-du) = -\int du = -u + C = -e^{\cos x} + C$

$I = -e^{\cos x} + C$.

Trigonometric functions:

1. Evaluate $\int \cos^3\theta \sin\theta \, d\theta$.

Q. **Solution:** Given: $\int \cos^3\theta \sin\theta \, d\theta$

Let $I = \int \cos^3\theta \sin\theta \, d\theta$

put $u = \cos\theta$, $du = -\sin\theta \, d\theta$.

$$I = \int u^3 (-du) = -\int u^3 du$$

$$= -\frac{u^4}{4} + C$$

$$I = -\frac{\cos^4\theta}{4} + C.$$

2. Evaluate $\int \sec^2\theta \tan^2\theta \, d\theta$.

Q **Solution:** Given: $\int \sec^2\theta \tan^2\theta \, d\theta$

Let $I = \int \sec^2\theta \tan^2\theta \, d\theta$.

put $u = \tan\theta \Rightarrow du = \sec^2\theta \, d\theta$

$$I = \int u^2 du = \frac{u^3}{3} + C$$

$$I = \frac{\tan^3\theta}{3} + C.$$

3. $\int \frac{\sec^2 x}{5+4\tan x} dx$

Q **Solution:** Given: $\int \frac{\sec^2 x}{5+4\tan x} dx$

Let $I = \int \frac{\sec^2 x}{5+4\tan x} dx$.

put $u = 5+4\tan x$ $\quad du = 4\sec^2 x \, dx$ $\quad \frac{du}{4} = \sec^2 x \, dx$.

$$I = \int \frac{1}{u} \frac{du}{4} = \frac{1}{4} \log u + C$$

$$I = \frac{1}{4} \log(5+4\tan x) + C.$$

(12)

4. Evaluate $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$.

(2)

Solution: Given: $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$.

Let $I = \int_0^{\pi/2} \cos x \sin(\sin x)\,dx$.

put $u = \sin x \Rightarrow du = \cos x\,dx$

$x \to 0 \Rightarrow u \to 0 \;;\; x \to \pi/2 \Rightarrow u \to 1$.

$\therefore I = \int_0^1 \sin u\,du = [-\cos u]_0^1 = (-\cos 1) - (-1)$

$I = 1 - \cos 1$.

5. Evaluate $\int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$.

(3)

Solution: Given: $I = \int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$.

Put $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}dx$

$\tan u = x \;;\; 1+x+x^2 = 1 + \tan u + \tan^2 u = \tan u + \sec^2 u$

$I = \int e^u [\tan u + \sec^2 u]\,du \quad [\because 1+\tan^2\theta = \sec^2\theta]$

put $t = e^u \tan u$, $dt = [e^u \sec^2 u + \tan u\, e^u]\,du$

$I = \int dt = t + C = [e^u \tan u] + C$

$I = x\, e^{\tan^{-1}x} + C$

6. H.W $\int \frac{\sin(\log x)}{x}dx$ Ans: $-\cos(\log x) + C$.

7. H.W $\int \frac{\sin 2x}{1+\cos^2 x}dx$ Ans: $-\log(1+\cos^2 x) + C$.

(4) (Q) 8. Evaluate $\int \frac{\tan x}{\sec x + \cos x}dx$.

U.Q Jan (2018)

Solution: $\int \frac{\tan x}{\sec x + \cos x}dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x}dx = \int \frac{\sin x}{1+\cos^2 x}dx$ $\quad u = \cos x$, $du = -\sin x\,dx$

$= \int \frac{-\cancel{du}}{1+u^2}du = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C$.

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3.3 Techniques of integration:

1) **Integration by parts:**

$$\int u\,dv = uv - \int v\,du$$

2) Integration by parts formula for definite integrals.

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

3) **Bernoulli's formula:**

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$v \to$ Integrating w.r.to x.

$u \to$ differentiating w.r.to x.

* Evaluate:

1. $\int x \sin x\,dx.$

Solution: Given: $\int x\sin x\,dx.$ $\quad [x(-\cos x) - (1)(-\sin x)] + c.$

$-x\cos x + \sin x + c.$

Let $u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = \int \sin x\,dx = -\cos x$

$$\int u\,dv = uv - \int v\,du.$$

$$\int x\sin x\,dx = (x)(-\cos x) - \int (-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c.$$

2. $\int_0^{1/2} x\cos \pi x\,dx.$

Solution: Given: $\int_0^{1/2} x\cos\pi x\,dx.$

Let $u = x \qquad dv = \cos\pi x\,dx$

$du = dx \qquad v = \int \cos\pi x\,dx = \frac{\sin\pi x}{\pi}$

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

$\left[x\frac{\sin\pi x}{\pi} - (1)\frac{-\cos\pi x}{\pi^2}\right]_0^{1/2}$

$\left[\frac{1}{2}\frac{\sin\pi/2}{\pi} + \frac{\cos\pi/2}{\pi}\right] - \left[0 + \frac{1}{\pi^2}\right]$

$\frac{1}{2\pi}(1) + 0 - \frac{1}{\pi^2}$

$\frac{1}{2\pi} - \frac{1}{\pi^2}$

$$\int_0^{1/2} x\cos\pi x\,dx = \left[x\frac{\sin\pi x}{\pi}\right]_0^{1/2} - \int_0^{1/2}\frac{\sin\pi x}{\pi}\,dx.$$ (13)

$$= \left(\frac{1}{2}\frac{\sin\pi/2}{\pi}\right) - (0) - \frac{1}{\pi}\int_0^{1/2}\sin\pi x\,dx \quad \left[\because \sin\tfrac{\pi}{2}=1\right]$$

$$= \frac{1}{2\pi} - \frac{1}{\pi}\left[-\frac{\cos\pi x}{\pi}\right]_0^{1/2} = \frac{1}{2\pi} + \frac{1}{\pi^2}\left[\cos\pi x\right]_0^{1/2}$$

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}\left[\cos\frac{\pi}{2} - \cos 0\right] \quad \left[\because \begin{array}{l}\cos\pi/2 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}\left[0-1\right] = \frac{1}{2\pi} - \frac{1}{\pi^2}$$

$$= \frac{\pi-2}{2\pi^2}.$$

3. Evaluate: $\int(\log x)^2\,dx$ [U.Q. Jan 2015]

(2) Solution: Given: $\int(\log x)^2\,dx$ $\quad u = \log x.$ $\quad du = \frac{1}{x}\,dx.$ $\quad = \int u^2.$

(Q) Let $u = (\log x)^2$ $\quad dv = dx$

$du = 2\log x\left(\frac{1}{x}\right)dx$ $\quad v = \int dx = x.$

$$\int u\,dv = uv - \int v\,du$$

$$\int(\log x)^2\,dx = (\log x)^2 x - \int x\,2\log x\left(\frac{1}{x}\right)dx$$

$$= x(\log x)^2 - 2\int\log x\,dx \quad \rightarrow (1)$$

Take, $\int\log x\,dx.$

Let $u = \log x$ $\quad dv = dx$

$du = \frac{1}{x}\,dx$ $\quad v = \int dx = x.$

$$\int u\,dv = uv - \int v\,du.$$

$$\int\log x\,dx = (\log x)(x) - \int x\frac{1}{x}\,dx = x\log x - \int dx$$

$$= x\log x - x.$$

$$(1) \Rightarrow \int(\log x)^2\,dx = x(\log x)^2 - 2\left[x\log x - x\right] + C.$$

H.W

4. $\int t \sin 2t \, dt$ Ans: $-\frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t + C.$

5. $\int_1^2 \log x \, dx$ Ans: $\log 4 - 1.$

6. $\int \frac{x}{1+\cos x} dx.$

(2) Solution: Given: $\int \frac{x}{1+\cos x} dx$

Let $\int \frac{x}{1+\cos x} dx = \int \frac{x}{2\cos^2 x/2} dx$ $[\because 1+\cos x = 2\cos^2 x/2]$

$= \frac{1}{2} \int x \sec^2 x/2 \, dx. \rightarrow (1)$

Let $u = x$ $dv = \sec^2 x/2 \, dx.$

$du = dx$ $v = \int \sec^2 x/2 \, dx = \frac{\tan(x/2)}{(1/2)} = 2\tan x/2.$

$\int u \, dv = uv - \int v \, du$

$(1) \Rightarrow \int \frac{x}{1+\cos x} dx = \frac{1}{2}\left[x(2\tan x/2) - \int 2\tan x/2 \, dx\right]$

$= x \tan x/2 - \frac{\log[\sec(x/2)]}{1/2} + C.$

$= x \tan x/2 - 2\log[\sec(x/2)] + C.$

7. H.W

$\int \frac{x}{1+\sin x} dx$

Ans: $\int \frac{x}{1+\sin x} dx = 2\tan x - \log(\sec x) - x \sec x$
$+ \log(\sec x + \tan x) + C.$

$\int \frac{x}{}$

8. Evaluate $\int t^2 e^t \, dt$

Solution: Given: $\int t^2 e^t \, dt$.

Let $u = t^2$, $u' = 2t$, $u'' = 2$, $u''' = 0$.

$v = e^t$, $v_1 = e^t$, $v_2 = e^t$, $v_3 = e^t$.

$$\int uv \, dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$$\int t^2 e^t \, dt = (t^2)(e^t) - (2t)(e^t) + 2(e^t) + c$$

$$\int t^2 e^t \, dt = (t^2 - 2t + 2)e^t + c.$$

$u = t^2$ $dv = e^t \, dt$.

$\frac{du}{dt} = 2t$, $v = e^t$

$\int u \, dv$

$= uv - \int v \, du$.

$= t^2 e^t - \int e^t . 2t \, dt$

$= t^2 e^t - 2\int e^t . t \, dt$

$= t^2 e^t - 2[te^t - \int e^t dt$

$= t^2 e^t - 2te^t + 2e^t + c$

$\int te^t dt$

$u = t$ $dv = e^t dt$

$du = dt$ $v = e^t$

H.W

9. Evaluate $\int x^5 e^x \, dx$

10. Evaluate $\int e^{ax} \cos bx \, dx$ using integration by parts.

Solution: Given: $\int e^{ax} \cos bx \, dx$.

Let $I = \int e^{ax} \cos bx \, dx$.

Let $u = e^{ax}$, $dv = \cos bx \, dx$

$du = e^{ax} . a \, dx$ $\quad v = \int \cos bx \, dx$

$= a e^{ax} dx$ $\quad v = \frac{\sin bx}{b}$

$$\therefore \int u \, dv = uv - \int v \, du$$

$$I = e^{ax} \frac{\sin bx}{b} - \int \frac{\sin bx}{b} e^{ax} . a \, dx$$

$$= \frac{e^{ax}}{b} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \rightarrow (1)$$

Take, $\int e^{ax} \sin bx \, dx$,

Let $u = e^{ax}$, $dv = \sin bx \, dx$

$du = e^{ax} . a \, dx$ $\quad v = \int \sin bx \, dx$

$= a e^{ax} dx$ $\quad v = -\frac{\cos bx}{b}$

$$\text{so } \int e^{ax} \sin bx \, dx = -e^{ax} \frac{\cos bx}{b} - \int \left(-\frac{\cos bx}{b} \right) e^{ax} . a \, dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}\int \cos bx \, e^{ax}\, dx$$

$$= -\frac{e^{ax}}{b}\cos bx + \frac{a}{b^2} I$$

$$① \rightarrow \quad I = \frac{e^{ax}}{b}\sin bx - \frac{a}{b}\left[-\frac{e^{ax}}{b}\cos bx + \frac{a}{b^2} I\right] + C_1$$

$$= \frac{e^{ax}}{b}\sin bx + \frac{a}{b^2} e^{ax}\cos bx - \frac{a^2}{b^2} I + C_1$$

$$I + \frac{a^2}{b^2} I = \frac{e^{ax}}{b}\sin bx + \frac{a}{b^2} e^{ax}\cos bx + C_1$$

$$I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{b}\left[\sin bx + \frac{a}{b}\cos bx\right] + C_1$$

$$I = \frac{e^{ax}}{\cancel{b}} \times \frac{b^{\cancel{2}}}{a^2+b^2}\left[\frac{b\sin bx + a\cos bx}{\cancel{b}}\right] + \frac{C_1 b^2}{a^2+b^2}$$

$$I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + C \quad \text{where } C = \frac{C_1 b^2}{a^2+b^2}$$

$$\therefore \int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + C.$$

H.W

11. Evaluate $\int e^x \sin x \, dx$

<u>Ans:</u> $I = \frac{1}{2} e^x(\cos x + \sin x) + C.$

12. Evaluate $\int e^x \cos x \, dx$

<u>Ans:</u> $I = \frac{1}{2} e^x(\sin x + \cos x) + C.$

13. Evaluate $\int \tan^{-1}x \, dx$. Also find $\int_0^1 \tan^{-1}x \, dx$.

<u>Solution:</u> <u>Given:</u> $\int \tan^{-1}x \, dx.$

Let $u = \tan^{-1}x$ $\quad dv = dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = \int dx = x.$

$$\int u\,dv = uv - \int v\,\cancel{dx}.$$

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$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx.$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2}dx. \rightarrow ①$$

Take, $\int \frac{x}{1+x^2}dx$,

put $t = 1+x^2 \Rightarrow dt = 2x\,dx$

$$\int \frac{x}{1+x^2}dx = \int \frac{1}{t}\frac{1}{2}dt = \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log t = \frac{1}{2}\log(1+x^2)$$

$$① \Rightarrow \int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C \rightarrow ②$$

To find $\int_0^1 \tan^{-1}x\,dx$

$$② \Rightarrow \int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \left[\tan^{-1}1 - 0\right] - \left[\frac{1}{2}\log 2 - \frac{1}{2}\log 1\right]$$

$$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}\log 2 \qquad [\because \log 1 = 0]$$

14. Evaluate $\int_0^{1/2} \cos^{-1}x\,dx$

Ans: $\int_0^{1/2} \cos^{-1}x\,dx = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$

15. Evaluate $\int \sin^{-1}x\,dx$

Ans: $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C.$