JACOBIANS:

Def:

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v w.r.to x, y. It is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left[\frac{\partial(u,v)}{\partial(x,y)}\right]$.

NOTE:

The Jacobian of u, v, w w.r.to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Properties of Jacobians:

i) If u and v are the functions of x and y then,

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \quad \text{or} \quad J, J' = 1.$$

where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ $\left[J' = \frac{1}{J}\right]$

ii) If u, v are the functions of x, y where and x, y are the functions of r, s then, $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(r,s)}$

iii) If u, v, w are functionally dependent functions of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

U.Q - 2010, 2011, 2013, 2014, 2016.

1. If $x = r\cos\theta$, $y = r\sin\theta$, find i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

(ii) ①

Sol:

Given: $x = r\cos\theta$, $y = r\sin\theta$ $\frac{\partial x}{\partial r} = \cos\theta$, $\partial y/\partial r = \sin\theta$

$\partial x/\partial\theta = -r\sin\theta$, $\frac{\partial y}{\partial\theta} = r\cos\theta$

W.K.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$r \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

$\frac{\partial(r,\theta)}{\partial(x,y)} = 1/r$.

$$= r\cos^2\theta + r\sin^2\theta = r\left[\cos^2\theta + \sin^2\theta\right] = r$$

2. If $x = u(1+v)$ and $y = v(1+u)$ find $\frac{\partial(x,y)}{\partial(u,v)}$

Jan 2011 H.W

①

Sol: Given: $x = u + uv$ $y = v + uv$

$$\frac{\partial x}{\partial u} = 1 + v \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1 + u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+u)(1+v) - uv = 1 + u + v + \cancel{uv} - \cancel{uv}$$

$$= 1 + u + v.$$

ⓧ H.W Jan - 2010, 2012.

② if $u = y^2/2x$, $V = \frac{x^2+y^2}{2x}$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Ans. $-y/2x$

3. If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

ⓧ ②

2001 2010 Jan

Sol: Given: $x = \frac{y^2}{x}$ $v = \frac{x^2}{y}$

$$\frac{\partial u}{\partial x} = -\frac{y^2}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{2x}{y}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} \qquad \frac{\partial v}{\partial y} = -x^2/y^2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & -x^2/y^2 \end{vmatrix}$$

$$= \frac{x^2y^2}{x^2y^2} - \frac{4xy}{xy} = 1 - 4 = -3.$$

4. Find the Jocobian of the transformation $x = r\cos\theta$, $y = r\sin\theta$.

Sol: $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

③

4. If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

(X) Sol: Given:

Nov-2001 2009 2011 Jan

$u = 2xy$ $\quad v = x^2 - y^2$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

$\frac{\partial u}{\partial x} = 2y$ $\quad \frac{\partial v}{\partial x} = 2x$ $\quad \frac{\partial x}{\partial r} = \cos\theta$ $\quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial u}{\partial y} = 2x$ $\quad \frac{\partial v}{\partial y} = -2y$ $\quad \frac{\partial x}{\partial \theta} = -r\sin\theta$ $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2)\, r(\cos^2\theta + \sin^2\theta)$$

$$= -4(x^2+y^2)\, r$$

$$= -4r\left[(r\cos\theta)^2 + (r\sin\theta)^2\right]$$

$$= -4r\left[r^2\cos^2\theta + r^2\sin^2\theta\right] = -4r.r^2(\sin^2\theta + \cos^2\theta)$$

$$= -4r^3$$

~~5~~. If $x = u(1-v), y = uv$, ~~then~~ find J and J' and P.T $JJ' = 1$.

H.W. ③ Sol:

Given: $x = u(1-v), y = uv$

$$\frac{\partial x}{\partial u} = 1-v \qquad \Bigg| \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = -u \qquad \Bigg| \qquad \frac{\partial y}{\partial v} = u.$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv = u - \cancel{vu} + \cancel{uv} = u$$

$$J = u.$$

Given:

$$x = u - uv \qquad y = uv \qquad x = u - y$$

$$\left[x + y = u - uv + uv = u\right] \qquad x + y = u$$

$$u = x + y \text{ and } v = \frac{y}{u} = \frac{y}{x+y}.$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2} \qquad \frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$$

$$\frac{(x+y).1 - y(1)}{(x+y)^2} \quad \frac{x + \cancel{y} - y}{(x+y)^2}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$J' = \frac{1}{u}$$

$$\boxed{\begin{aligned} J &= u \\ J' &= 1/u \\ \therefore JJ' &= \cancel{u} \times \frac{1}{\cancel{u}} \\ JJ' &= 1 \end{aligned}}$$

ⓧ U.Q 2010 2014 2008 2015 2016 2017

4. Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Solution:

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-\cancel{x_2}\cancel{x_3}}{\cancel{x_1^2}}\left[\frac{x_1^2 \cancel{x_2}\cancel{x_3}}{\cancel{x_2^2}\cancel{x_3^2}} - \frac{x_1^2}{x_2 x_3}\right] - \left[\frac{x_3}{x_1}\right]$$

$$\left[\frac{-x_1}{x_3} - \frac{x_1 x_2}{x_2 x_3}\right] + \frac{x_2}{x_1}\left[\frac{x_1}{x_2} + \frac{x_1}{x_2}\right]$$

$$= -1 + 1 + 1 + 1 + 1 + 1$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$$

2011 2016 2015 2009

5. Find the Jacobian $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ of the transformation $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$.

Solution: The Jacobian of transformation,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}.$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta\cos\phi$ $\quad$ $y = r\sin\theta\sin\phi$ $\quad$ $z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\quad$ $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\quad$ $\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ $\quad$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ $\quad$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ $\quad$ $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$ $\quad$ $\frac{\partial z}{\partial \phi} = 0.$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\theta\cos\phi$$

$$\left[0 - r\sin\theta\cos\theta\cos\phi\right] + \left(-r\sin\theta\sin\phi\right)$$

$$\left[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right]$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin\theta\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right)$$

$$= r^3\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right]$$

$$J = r^2\sin\theta.$$

6. if $u = x-y$, $v = y-z$, $w = z-x$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

2006

Solution:

Given: $u = x-y$, $v = y-z$, $w = z-x$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 1 + 1(-1)$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Jan 2011
Jan-2013
Jan-2014
D-2015
J-2016

7. if $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Solution: Given: $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$.

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{z}{y} \qquad \frac{\partial w}{\partial x} = y/z$$

$$\frac{\partial u}{\partial y} = \frac{z}{x} \qquad \frac{\partial v}{\partial y} = -\frac{zx}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{x} \qquad \frac{\partial v}{\partial z} = \frac{x}{y} \qquad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left(\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(\frac{-xy}{yz} - \frac{x}{z}\right) + y/x\left(x/y + x/y\right)$$

$$= -1+1+1+1+1+1$$

$$= 4$$

6. Prove $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$ (12) are functionally dependent. Find the relationship between them.

Sol: Given: $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 1\left[2z(z+x) - 2y(x+y)\right] - 1\left[2z(y+z) - 2x(x+y\right.$$
$$+ 1\left[2y(y+z) - 2x(z+x)\right]$$

$$= \cancel{2z^2} + \cancel{2zx} - \cancel{2yx} - \cancel{2y^2} - \cancel{2zy} - \cancel{2z^2} + \cancel{2x^2}$$
$$+ \cancel{2xy} + \cancel{2y^2} + \cancel{2yz} - \cancel{2xz} - \cancel{2x^2}$$

$J = 0$ $\therefore$ u and v are not independent.

$\therefore$ u, v and w are functionally dependent.

The relation between is given by the formula,

$$(x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx)$$

$$u^2 = w + 2v.$$

Method of Lagrangian Multiplier:-

To find the values of x, y, z for which the maximum and minimum values of $f(x, y, z)$ can have a conditional equation $g(x, y, z) = 0$, then the auxiliary function $F(x, y, z)$ given by,

$$F(x, y, z) = f(x, y, z) + \lambda\, g(x, y, z) \rightarrow ①$$

where λ is called "Lagrange Multiplier" which is independent of x, y, z.

To find $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$, $\frac{\partial F}{\partial \lambda} = 0 \rightarrow ②$

using ② we can solve for x, y and z.

1. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Sol

Let $f = x^2 + y^2 + z^2$

$g = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$

Let the auxiliary function 'F' be

$$F(x, y, z) = f(x, y, z) + \lambda\, g(x, y, z)$$

$$= (x^2 + y^2 + z^2) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right) \rightarrow ①$$

where λ is Lagrange multiplier.

$\frac{\partial F}{\partial x} = 2x + \lambda\left(-\frac{1}{x^2}\right) = 2x - \lambda/_{x^2}$, $\frac{\partial F}{\partial y} = 2y - \frac{\lambda}{y^2}$, $\frac{\partial F}{\partial z} = 2z - \frac{\lambda}{z^2}$

$\frac{\partial F}{\partial \lambda} = \frac{1}{x} + \frac{1}{y} + 1/z - 1$

For a minimum at (x, y, z) we have,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial F}{\partial x} = 2x + \lambda\left(\frac{-1}{x^2}\right) = 2x - \frac{\lambda}{x^2} = 0$$

$$\Rightarrow 2x^3 - \lambda = 0$$

$$\Rightarrow 2x^3 = \lambda$$

$$\Rightarrow x^3 = \lambda/2$$

$$\Rightarrow x = (\lambda/2)^{1/3} \rightarrow ②.$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y - \frac{\lambda}{y^2} = 0 \Rightarrow y^3 = \lambda/2 \Rightarrow y = (\lambda/2)^{1/3}$$

$$y = (\lambda/2)^{1/3} \rightarrow ③.$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z - \frac{\lambda}{z^2} = 0 \Rightarrow z^3 = \lambda/2 \qquad \frac{\partial F}{\partial \lambda} = 0$$

$$z = (\lambda/2)^{1/3} \rightarrow (4). \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \rightarrow (5$$

From ② & ③ weget,

$$x = y \rightarrow (6)$$

From ③ & (4) weget,

$$y = z \rightarrow (7).$$

From (6) & (7) weget,

$$x = y = z.$$

From (5) $\Rightarrow$ $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = 1$

$$\Rightarrow \frac{3}{x} = 1$$

$$\Rightarrow x = 3$$

$$\Rightarrow y = 3 \text{ \& } z = 3.$$

$\therefore$ $(3, 3, 3)$ is the point where minimum value occurs

• The minimum value is, $x^2+y^2+z^2 = 3^2+3^2+3^2 = 9+9+9 = 27.$

2011, Jan2014, 2008, 2016. Ans : 3(2k)$^{2/3}$.

2. A rectangular H.W ① (k) box open at the top, is to have a volume of 32cc. Find the dimensions of the box, that requires the least Material for its construction.

2002 Jan2010 Dec2010 2014 2016 2017 ①

NOTE: The surface area of the rectangular box is = ~~2xy~~+2yz+2zx

Sol: Let x, y, z be the length, breadth and height of the box.

16 + 16 + 16

$(4)(4) + 2(4)(2) + 2(2)(4)$

$\therefore$ The surface area $S = xy + 2yz + 2zx \to Ⓐ$ and

Volume $V = xyz = 32 \to Ⓑ$.

Let the auxiliary function F be. 48

$$F(x, y, z) = f + \lambda g$$

$$= (xy + 2yz + 2zx) + \lambda(xyz - 32) \to ①.$$

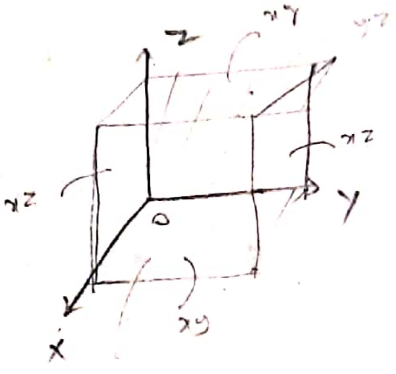
$$= xy + 2yz + 2zx + \lambda xyz - 32\lambda$$

where λ is Lagrange multiplier.

$$\frac{\partial F}{\partial x} = y + 2z + \lambda yz \qquad \frac{\partial F}{\partial \lambda} = xyz - 32.$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx$$

$$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy.$$



where F is extremum.

$$\frac{\partial F}{\partial x} = 0$$

$y + 2z + \lambda yz = 0 \Rightarrow$ ~~xyz~~ $y + 2z = -\lambda yz$

$\div yz$ $\frac{1}{z} + \frac{2}{y} = -\lambda$

$\Rightarrow \lambda yz = -y - 2z$

$\Rightarrow \lambda = -\frac{\cancel{y}}{\cancel{y}z} - \frac{2z}{yz} = -\frac{1}{z} - \frac{2}{y}$

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$$\frac{1}{z} + \frac{2}{y} = -\lambda \quad \rightarrow ②.$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \quad x + 2z + \lambda zx = 0$$

$$x + 2z = -\lambda zx$$

$\div zx,$

$$\frac{1}{z} + \frac{2}{x} = -\lambda \quad \rightarrow ③.$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \quad 2x + 2y + \lambda xy = 0.$$

$$2x + 2y = -\lambda xy$$

$\div xy,$

$$\frac{2}{y} + \frac{2}{x} = -\lambda \quad \longrightarrow ④.$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32 = 0$$

$$xyz = 32 \rightarrow \circledast$$

From ② and ③ weget

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\Rightarrow \frac{2}{y} = \frac{2}{x}$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y \rightarrow ⑤$$

From ③ and ④ weget,

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$\Rightarrow \frac{1}{z} = \frac{2}{y}$$

$$\Rightarrow y = 2z \rightarrow (6).$$

From ⑤ & ⑥ weget,

$$x = y = 2z.$$

⑦ volume $\circledast \Rightarrow xyz = 32$

$$(2z)(2z)\,z = 32$$

$$4z^3 = 32$$

$$z^3 = \frac{32}{4} = 8$$

$$z = 2,$$

$$x = 2z = 2.2 = 4$$

$$x = 4$$

$$y = 2z, \quad y = 4$$

$$x = y = 2(k/4)^{1/3}$$

$$\text{Minimum} = xy + 2yz + 2zx.$$

$$xyz = k$$

$$(k/4)^{1/3}(k/4)^{1/3} 4z^3 = k$$

$$z^3 = (k/4)^{1/3}$$

$$z = (k/4)$$

$$= 2(k/4)^{1/3} 2(k/4)^{1/3} + 2\,2(k/4)^{1/3}(k/4)^{1/3} + 2.2(k/4)^{1/3}(k/4)^{1/3}$$

$$= 4(k/4)^{2/3} + 4(k/4)^{2/3} + 4(k/4)^{2/3}$$

$$= 12(k/4)^{2/3}$$

The minimum cost. $x = 4, \; y = 4, \; z = 2.$

The dimension of the box are $4, 4, 2$.

3. A thin closed rectangular box is to have ㉖ one edge equal to twice the other, and a constant volume $72\,m^3$. Find the least surface area of the box.

H.W ②

Sol: Let $x, 2x, y$ be the length, breadth and height of the box, respectively.

$$\overset{2xy \quad + 2yz \quad + 2zx.}{\text{surface area} = 2(x \cdot 2x) + 2(2x \cdot y) + 2(y \cdot x)}$$

$$= 4x^2 + 4xy + 2xy$$

$$f = 4x^2 + 6xy \rightarrow Ⓐ$$

and volume $xyz = 72$

$$\text{ie, } xy(2x) = 72$$

$$\Rightarrow 2x^2y = 72$$

$$\Rightarrow x^2y = 36 = g \rightarrow Ⓑ$$

Let F be the auxiliary function,

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$= (4x^2 + 6xy) + \lambda(x^2y - 36)$$

$$= 4x^2 + 6xy + \lambda x^2y - \lambda 36 \rightarrow ①.$$

where λ is L.M.

$$\frac{\partial F}{\partial x} = 8x + 6y + 2\lambda xy$$

$$\frac{\partial F}{\partial y} = 6x + \lambda x^2$$

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2y - 36$$

When F is extremum.

$\frac{\partial F}{\partial x} = 0$ $\qquad$ $\frac{\partial F}{\partial y} = 0$

$\Rightarrow 8x + 6y + 2\lambda xy = 0. \rightarrow$ ② $\qquad \Rightarrow 6x + \lambda x^2 = 0 \rightarrow$ ③

$\frac{\partial F}{\partial x} = 0$ $\qquad$ $\frac{\partial F}{\partial \lambda} = 0$

$\Rightarrow 0 = 0$ $\qquad x^2y - 36 = 0 \Rightarrow x^2y = 36 \rightarrow$ ④.

From ③ weget,

$$6\cancel{x} = -\lambda x^{\cancel{2}}$$

$$x = -\frac{6}{\lambda} \rightarrow (5) \qquad -\lambda = 6/x. \rightarrow ⑤$$

Sub's $x = -6/\lambda$ in ④ weget,

$8x + 6y + 2\lambda xy.$

$8x + 6y = -2\lambda xy$

$$x^2y = 36$$

$$\Rightarrow y = \frac{36}{x^2} = \frac{36}{36}\lambda^2 = \lambda^2$$

$\frac{8x}{2xy} + \frac{6y}{2xy} = -\lambda.$

$$\Rightarrow y = \lambda^2 \rightarrow (6)$$

$\frac{4}{y} + \frac{3}{x} = -\lambda \rightarrow ⑥$

Sub's $x = -6/\lambda$ and $y = \lambda^2$ in ② weget,

$$8\times\frac{-6}{\lambda} + 6\times\lambda^2 + 2\times\frac{-6}{\lambda}\lambda^2 = 0$$

From ⑤ & ⑥,

$\frac{6}{x} = \frac{4}{y} + \frac{3}{x}.$

$\frac{6}{x} - \frac{3}{x} = \frac{4}{y}$

$\frac{3}{x} = \frac{4}{y}$

$$-\frac{48}{\lambda} + 6\lambda^2 - 12\lambda^2 = 0.$$

$$~~48~~ \; -\frac{48}{\lambda} + (-6\lambda^2) = 0$$

$$-\frac{48}{\lambda} - 6\lambda^2 = 0$$

$x^2y = 36$

$3y = 4x.$

$\left(\frac{3y}{4}\right)^2 y = 36.$

$\Rightarrow x = \frac{3y}{4}$

$\frac{9y^2}{16}y = 36$

$\frac{y^3}{16} = 4$

$y^3 = 64 \quad y = 4$

$x = 3. \quad x = \frac{3}{4}(4)$

$$6\lambda^3 + 48 = 0$$

$$6\lambda^8 = -48 \Rightarrow \lambda^3 = -\frac{48}{6} = -8$$

$$\lambda^3 = -8$$

$$\lambda = -2$$

sub's $\lambda = -2$ in (5) & (6),

$$x = \frac{-6}{\lambda} = \frac{-6}{-2} = 3$$

$$\boxed{x = 3}$$

$$y = \lambda^2 = (-2)^2 = 4$$

$$\boxed{y = 4}$$

$\therefore$ f is minimum at (3, 4).

$\therefore$ The minimum value of $f = 6xy + 4x^2$

$$= 6(3)(4) + 4(3)^2$$

$$= 6(12) + 4(9)$$

$$= 72 + 36$$

$$= 108 \,//.$$

Q4. Find the volume of the largest rectangular solid which can be inscribed in the ellipsoid

1998
2009
2015

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

②

Sol Let the edges of the parallelopiped be $2x, 2y \cdot 2z$

$\therefore$ volume $V = 2x \cdot 2y \cdot 2z$

$$= 8xyz$$

$$f = 8xyz.$$

Now we have to find the Maximum value of the volume V subject to the condition that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

ie., $$g=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

The auxiliary function 'F' is given by,

$$F(x,y,z)=f+\lambda g$$

ie, $$F(x,y,z)=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\rightarrow ①.$$

$$\frac{\partial F}{\partial x}=8yz+\frac{2x\lambda}{a^2}$$

$$\frac{\partial F}{\partial y}=8xz+\lambda.\frac{2y}{b^2}$$

$$\frac{\partial F}{\partial z}=8xy+\lambda.\frac{2z}{c^2}$$

$$\frac{\partial F}{\partial \lambda}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1.$$

② $8yz=-\frac{\lambda 2x}{a^2}$ ③ $8xz=-\frac{\lambda 2y}{b^2}$

$\frac{\cancel{8}yza^2}{\cancel{2}x}=-\lambda$ $\frac{8xz\,b^2}{\cancel{2}y}=-\lambda$

$\frac{4yza^2}{x}=-\lambda$ $\frac{4xzb^2}{y}=-\lambda$

$\frac{\cancel{4}y\cancel{z}a^2}{x}=\frac{\cancel{4}x\cancel{z}b^2}{y}$ $\frac{4xyc^2}{z}=-\lambda$

$\frac{y}{x}a^2=\frac{xb^2}{y}$

$y^2a^2=x^2b^2\Rightarrow\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$

Since F is extremum,

$$\frac{\partial F}{\partial x}=0\Rightarrow 8yz+\frac{\lambda 2x}{a^2}=0\rightarrow (2)$$

(2)×x, $8xyz+\frac{\lambda 2x^2}{a^2}=0$

$$\frac{\partial F}{\partial y}=0\Rightarrow 8xz+\lambda\frac{2y}{b^2}=0\longrightarrow ③$$

(3)×y, $8xyz+\frac{\lambda 2y^2}{b^2}=0.$

$$\frac{\partial F}{\partial z}=0\rightarrow 8xy+\lambda.\frac{2z}{c^2}=0\longrightarrow (4)$$

(4)×z, $8xyz+\frac{\lambda 2z^2}{c^2}=0.$

$$\frac{\partial F}{\partial \lambda}=0\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0\rightarrow(5)$$

Adding ③, ④ & ⑤ weget,

~~24xyz~~

$$8yzx + 8zxy + 8xyz + \frac{2\lambda x^2}{a^2} + \frac{2\lambda y^2}{b^2} + \frac{2\lambda z^2}{c^2} = 0.$$

$$24xyz + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

i.e., $24xyz + 2\lambda = 0.$ $\left[\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2/c^2 = 1\right]$

$\Rightarrow 24xyz = -2\lambda$

$\Rightarrow 12xyz = -\lambda$

$\Rightarrow \lambda = -12xyz$ in ② weget,

$$8yz - 12xyz\,\frac{2x}{a^2} = 0$$

$$-\frac{\overset{3}{\cancel{24}}x^2 yz}{a^2} = -\cancel{8yz}$$

$$3x^2 = a^2$$

$$x^2 = a^2/3$$

$$x = a/\sqrt{3}$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\frac{x^2}{a^2} + x^2/a^2 + x^2/a^2 = 1$

$\frac{3x^2}{a^2} = 1 \Rightarrow 3x^2 = a^2$

$x^2 = \frac{a^2}{3}$

$x = \frac{a}{\sqrt{3}}$

Similarly, $y = b/\sqrt{3}$ and $z = \cancel{a}/\sqrt{3} = c/\sqrt{3}$. $y = b/\sqrt{3}$, $z = c/\sqrt{3}$.

$\therefore$ At $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ the volume 'V' is maximum.

The maximum value of $V = 8xyz$

$$= 8\,\frac{a}{\sqrt{3}}\,\frac{b}{\sqrt{3}}\,\frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}}.$$

5. Find the Maximum value of $x^m y^n z^p$, when $x+y+z=a$.

Sol: Let $f = x^m y^n z^p$ — Ⓐ and $g = x+y+z-a$ — Ⓑ

⑤

Jan-2008
Jan-2011
Dec-2011

Let 'F' be the auxiliary function.

$$F = f + \lambda g$$

$$F(x,y,z) = x^m y^n z^p + \lambda(x+y+z-a) \rightarrow ①.$$

③

$$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda = 0 \rightarrow ②$$

$$\frac{\partial F}{\partial y} = n y^{n-1} x^m z^p + \lambda = 0 \rightarrow ③.$$

$$\frac{\partial F}{\partial z} = p z^{p-1} x^m y^n + \lambda = 0 \rightarrow ④$$

$$\frac{\partial F}{\partial \lambda} = x + y + z - a = 0 \rightarrow ⑤.$$

From ②, ③ and ④ we get,

$$-\lambda = m x^{m-1} y^n z^p$$

$$-\lambda = n x^m y^{n-1} z^p$$

$$-\lambda = p x^m y^n z^{p-1}$$

ie, $m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$

$* x^{-m} y^{-n} z^{-p} \Rightarrow m x^m x^{-m} x^{-1} y^n y^{-n} z^{-p} z^p = n x^m x^{-m} y^n y^{-n} y^{-1} z^p z^{-p}$

$= p x^m x^{-m} y^n y^{-n} z\, z^p z^{-p}$

$$\Rightarrow m x^{-1} = n y^{-1} = p z^{-1}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$= \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a} \quad [\text{by } Ⓑ]$$

∴ The Maximum value of f occurs when

$$\frac{m}{x} = \frac{m+n+p}{a}$$

$$x = \frac{am}{m+n+p}$$

ly, $$y = \frac{an}{m+n+p}$$

$$z = \frac{ap}{m+n+p}$$

The Maximum value of $f = x^m y^n z^p$

$$= \frac{a^m m^m}{(m+n+p)^m} + \frac{a^n n^n}{(m+n+p)^n} + \frac{a^p p^p}{(m+n+p)^p}$$

$$= \frac{a^{m+n+p} \; m^m \; n^n \; p^p}{(m+n+p)^{m+n+p}}$$

ⓧ 2005 2010 ④ 6. The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$.

Sol: Given: $T = f(x, y, z) = 400xyz^2 \rightarrow$ Ⓐ

$\phi(x, y, z) = x^2+y^2+z^2-1 \rightarrow$ Ⓑ

Let F be the auxiliary function,

$F = f + \lambda\phi \Rightarrow F = 400xyz^2 + \lambda(x^2+y^2+z^2-1) \rightarrow$ ①

Now, $\frac{\partial F}{\partial x} = 400yz^2 + \lambda 2x = 0 \rightarrow$ ②

$\frac{\partial F}{\partial y} = 400xz^2 + \lambda 2y = 0 \rightarrow$ ③

$= 800xyz + \lambda 2z = 0 \rightarrow$ ④

∂F

$$(2) \Rightarrow \quad 400yz^2 + \lambda 2x = 0$$

$$\Rightarrow \quad \lambda 2x = -400yz^2$$

$$\Rightarrow \quad \lambda = -\frac{200yz^2}{x} \rightarrow ⑤$$

$$(3) \Rightarrow \quad \lambda = -\frac{200xz^2}{y} \rightarrow ⑥$$

$$(4) \Rightarrow \quad \lambda = -400xy \rightarrow ⑦$$

From ⑤ & ⑥ weget,

$$\frac{200yz^2}{x} = \frac{200xz^2}{y}$$

$$y^2 = x^2 \rightarrow ⑧$$

From ⑥ & ⑦ weget.

$$\frac{200\cancel{x}z^2}{y} = \overset{2}{400}\cancel{x}y$$

$$z^2 = 2y^2 \rightarrow ⑨$$

$$\Rightarrow y^2 = z^2/2$$

From ⑧ & ⑨ weget,

$$x^2 = y^2 = z^2/2$$

From (B) $\Rightarrow \quad x^2 + y^2 + z^2 = 1$

$$\frac{z^2}{2} + \frac{z^2}{2} + \frac{z^2}{\cancel{2}} = 1 \quad \Rightarrow z^2\left[\frac{1}{2} + \frac{1}{2} + 1\right] = 1$$

$$\Rightarrow z^2[2] = 1 \quad \Rightarrow z^2 = 1/2$$

$$\Rightarrow z = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \quad x^2 = y^2 = \frac{z^2}{2} = \frac{1/2}{2} = 1/4$$

$$\Rightarrow x = \pm 1/2$$

$$y^2 = x^2 = 1/4$$

$$\Rightarrow y = \pm 1/2.$$

$$x = \pm 1/2, \quad y = \pm 1/2, \quad z = \pm \frac{1}{\sqrt{2}}.$$

The Maximum Temperature $T = 400xyz^2$

$$= \overset{\overset{100}{200}}{\cancel{400}}\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 50$$

∴ The highest temperature is 50°C

5. Find the dimensions of the rectangular box without a top of maximum capacity, whose surface is 108 sq. cm.

Solution: Let x, y, z be the length, breadh and height of the box.

$\therefore$ The surface area $\phi = xy + 2yz + 2zx = 108 \rightarrow Ⓐ$

Volume $V = xyz$

Let the auxiliary function F be,

$F(x,y,z,\lambda) = xyz + \lambda(xy + 2yz + 2zx - 108) \rightarrow Ⓑ$

where λ is Lagrange Multiplier.

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z)$$

$$\frac{\partial F}{\partial y} = zx + \lambda(x+2z)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2x+2y)$$

To find the stationary points,

$F_x = 0$

$\Rightarrow yz + \lambda(y+2z) = 0$

$yz = -\lambda(y+2z).$

$$\frac{yz}{(y+2z)} = -\lambda \rightarrow ①$$

$F_y = 0$

$\Rightarrow zx + \lambda(x+2z) = 0$

$zx = -\lambda(x+2z)$

$$\frac{zx}{x+2z} = -\lambda \rightarrow ②$$

$F_z = 0$

$xy + \lambda(2x+2y) = 0$

$xy = -\lambda(2x+2y).$

$$\frac{xy}{2x+2y} = -\lambda \rightarrow ③.$$

From ①, ② ③ weget

$$\frac{yz}{y+2z} = \frac{xz}{x+2z} = \frac{xy}{2(x+y)}$$

Taking 1st and 2nd ratio weget,

$$\frac{yz}{y+2z} = \frac{xz}{x+2z} \Rightarrow y(x+2z) = x(y+2z).$$

$$\Rightarrow \cancel{xy} + 2yz = \cancel{xy} + 2xz \Rightarrow 2y\cancel{z} = 2x\cancel{z}$$

$$\therefore x = y \rightarrow ④$$

Taking 2nd and 3rd ratio weget,

$$\frac{\cancel{x}z}{x+2z} = \frac{\cancel{x}y}{2(x+y)} \Rightarrow z(2x+2y) = y(x+2z)$$

$$\Rightarrow 2xz + \cancel{2yz} = yx + \cancel{2zy} \Rightarrow 2xz = yx.$$

$$z = \frac{y\cancel{x}}{2\cancel{x}} \Rightarrow z = y/2 \rightarrow ⑤$$

sub's ⑤ in ④ $z = y/2 = x/2 \rightarrow ⑥$.

sub's ④, ⑤, ⑥ in Ⓐ weget,

$$\phi = xy + 2xz + 2yz = 108.$$

$$\Rightarrow x(x) + \cancel{2}x.\frac{x}{\cancel{2}} + \cancel{2}x.\frac{x}{\cancel{2}} = 108$$

$$x^2 + x^2 + x^2 = 108 \Rightarrow 3x^2 = 108 \Rightarrow x^2 = 36.$$

$$\therefore x = 6.$$

$$\therefore y = 6,\ z = 6/2 = 3.$$

The dimensions of the box are, 6, 6, 3.

Length = 6 cm, breadth = 6cm, height = 3cm.

∴ Maximum volume = 6×6×3 = 108 cubic Metres.

H.W 6. Find the dimensions of the rectangular box without top of Maximum capacity with surface area 432 square Metre.

Ans: $x = 12$, $y = 12$, $z = 6$, $V = 864$ cm.

Maxima and minima for functions of two variables:

Let $f(x,y)$ be the given function. To find the maximum and minimum values of $f(x,y)$ we have to following the rules.

i) Find the partial derivatives of f, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ from $f(x,y)$.

ii) Solve $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

iii) Calculate the value of $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

iv) If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2>0$ and $\frac{\partial^2 f}{\partial x^2}<0$ or $\frac{\partial^2 f}{\partial y^2}<0$ then 'f' has a maximum value.

v) If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2>0$ and $\frac{\partial^2 f}{\partial x^2}>0$ or $\frac{\partial^2 f}{\partial y^2}>0$. then 'f' has a minimum value.

vi) If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2<0$ and $\frac{\partial^2 f}{\partial x^2}<0$ or $\frac{\partial^2 f}{\partial y^2}>0$. then 'f' has neither a maximum nor a minimum Such a point is called a saddle point.

vii) If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2=0$, then 'f' has a inconclusive

viii) Extremum value:

$f(x,y)$ is said to be extremum, if it is either a maximum or minimum.

① H.W. Discuss the Maximum and minimum of $x^2+y^2+6x+12$.

Sol : Given $f(x,y) = x^2+y^2+6x+12$.

$\frac{\partial f}{\partial x} = 2x+6$ | $\frac{\partial f}{\partial y} = 2y$ | $\frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial^2 f}{\partial x^2} = 2$ | $\frac{\partial^2 f}{\partial y^2} = 2$

To find the stationary points,

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$2x+6 = 0$ $\qquad$ $2y = 0$

$2x = -6$ $\qquad$ $y = 0$.

$x = -3$

∴ The stationary points are $(-3,0)$

At $(-3,0)$,

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(2) - 0 = 4 > 0$$

and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ $\quad$ $\frac{\partial^2 f}{\partial y^2} = 2 > 0$

∴ The point $(-3,0)$ is a minimum point.

∴ The minimum value is, $f(x,y) = x^2+y^2+6x+12$

$$f(x,y) = f(-3,0) = (-3)^2 + 0^2 + 6(-3) + 12$$

$$= 9 - 18 + 12$$

$$= 3.$$

1. A flat circular plate is heated so that the temperature at any point (x, y) is $u(x,y) = x^2+2y^2-x$. find the coldest point on the plate. (*) 2005

① **Solution:** Given: $u = x^2+2y^2-x$.

$u_x = 2x-1 \qquad u_y = 4y \qquad B = u_{xy} = 0.$

$A = u_{xx} = 2 \qquad C = u_{yy} = 4$

To find the stationary points,

$u_x = 0 \qquad u_y = 0$

$\Rightarrow 2x-1 = 0 \qquad 4y = 0$

$x = \frac{1}{2} \qquad y = 0.$

$\therefore$ ~~u is minimum~~ The s.p is $(\frac{1}{2}, 0)$.

$AC - B^2 = (2)(4) - 0 = 8 > 0$

$\therefore$ The point $(\frac{1}{2}, 0)$ is a minimum point.

$\therefore$ The minimum value is,

$f(x,y) = x^2+2y^2-x$

$= (\frac{1}{2})^2 + 2(0) - \frac{1}{2} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$f(x,y) = -\frac{1}{4}$

$\therefore$ The minimum value $= -\frac{1}{4}$.

2. Find the Maxima and Minima of $x^4+y^4-2x^2+4xy-2y^2$.

Jan 2009
Dec 1997
Jan. 2010

Solution: Given: $u(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

$U_x = 4x^3 - 4x + 4y$ $\quad$ $U_y = 4y^3 + 4x - 4y$

$A = U_{xx} = 12x^2 - 4$, $B = U_{xy} = 4$, $C = U_{yy} = 12y^2 - 4$.

To find the stationary points are,

$U_x = 0$ $\quad$ $U_y = 0$.

$4x^3 - 4x + 4y = 0$ $\quad$ $4y^3 + 4x - 4y = 0$

$x^3 - x + y = 0 \rightarrow ①$ $\quad$ $y^3 + x - y = 0 \rightarrow ②$

$① + ② \Rightarrow x^3 + y^3 = 0 \Rightarrow x^3 = -y^3 \Rightarrow x = -y$.

$① \Rightarrow x^3 - x - x = 0 \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$

$x = 0 \quad x = \pm\sqrt{2}$.

$\therefore$ The s.p are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

At $(0,0)$:

$A = 12x^2 - 4 = -4 < 0$

$C = 12y^2 - 4 = -4 < 0$

$B = 4$

$AC - B^2 = (-4)(-4) - (4)^2 = 0$

$\therefore$ The point $(0,0)$ cannot be an extreme point.

At $(\sqrt{2}, -\sqrt{2})$,

$A = 24 - 4 = 20 > 0$

$C = 24 - 4 = 20 > 0$

$B = 4$

$AC - B^2 = (20)(20) - (4)^2 = 384 > 0$.

$\therefore (\sqrt{2}, -\sqrt{2})$ is minimum point.

At $(-\sqrt{2}, \sqrt{2})$,

$A = 24 - 4 = 20 > 0$ $\quad$ $B = 4$

$C = 24 - 4 = 20 > 0$

$AC - B^2 = (20)(20) - 4^2 = 384 > 0$

$\therefore (\sqrt{2}, -\sqrt{2})$ is minimum point.

(i) minimum of $(\sqrt{2},$ [illegible]$2)$

$u(x,y) = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$

$= 4 + 4 - 4 - 8 - 4$

$= -8$.

(ii) minimum at $(-\sqrt{2}, \sqrt{2}$

$u(x,y) = -8$.

$\therefore (\sqrt{2}, -\sqrt{2}) (-\sqrt{2}, \sqrt{2})$ are point of minimum value and M.V is -8.

(19)

② 2: Find the maximum or minimum values of x^2-xy+y^2-2x+y.

2004
2010
2012

Sol Given $f(x,y) = x^2-xy+y^2-2x+y$.

$\frac{\partial f}{\partial x} = 2x-y-2$

$\frac{\partial^2 f}{\partial x^2} = 2$

$\frac{\partial f}{\partial y} = -x+2y+1$

$\frac{\partial^2 f}{\partial y^2} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = -1$

To find the stationary points,

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 2x-y-2=0$ $\qquad$ $-x+2y+1=0$

$\Rightarrow 2x-y=2 \rightarrow ①$ $\qquad$ $-x+2y=-1 \longrightarrow ②$.

Solving ① & ② we get,

$$\begin{aligned} ①\Rightarrow \quad & \cancel{2x}-y = 2 \\ ②\times 2 \Rightarrow \quad & -\cancel{2x}+4y = -2 \\ \hline & 3y = 0 \\ & \boxed{y = 0} \end{aligned}$$

sub $y=0$ in ① we get,

$2x-y=2$

$2x-0=2$

$2x=2$

$\boxed{x=1}$

$\therefore$ The stationary points are $(1,0)$.

At point $(1,0)$

$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(2)-(-1)^2$

$= 4-1 = 3>0$

and $\frac{\partial^2 f}{\partial x^2}=2>0$ or $\frac{\partial^2 f}{\partial y^2}=2>0$.

$\therefore$ The point $(1,0)$ is a minimum point.

The minimum value is,

$f(x,y) = x^2-xy+y^2-2x+y$

$f(1,0) = 1-0+0-2+0$

$f(1,0) = -1$

Scanned by CamScanner

5. Examine $f(x,y) = x^3+y^3-12x-3y+20$ for its extreme values. ②

H.W ② $f(x,y) = x^3+y^3-3x-12y+20$ Jan 2005 2010, 2012 2014 Dec 2010.

2002
2012

Sol : Given: $f(x,y) = x^3+y^3-12x-3y+20$

$$\frac{\partial f}{\partial x} = 3x^2-12 \qquad f_x = 3x^2-3$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad A = 6x$$

$$\frac{\partial f}{\partial y} = 3y^2-3 \qquad f_y = 3y^2-12.$$

$$\frac{\partial^2 f}{\partial y^2} = 6y \qquad B = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$C = 0.$

To find the stationary points, $C = f_{yy}$ $B = f_{xy}$ $A = f_{xx}$ point

$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

$$3x^2-12 = 0 \qquad\qquad 3y^2-3 = 0$$

$$3x^2 = 12 \qquad\qquad 3y^2 = 3$$

$$x^2 = 4 \qquad\qquad y^2 = 1$$

$$\Rightarrow x = \pm 2 \qquad 2,-1 \quad -1,-1 \qquad y = \pm 1$$

$x = \pm 1$ $\qquad$ $y = \pm 2.$

$AC-B^2$ $\quad f$

∴ The stationary points are (2, 1) (2, −1), (−2, 1), (−2, −1).

(1, 2) (1, −2) (−1, 2) (−1, −2)

At point (2, 1),

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (6\times 2).(6\times 1) - 0$$

$$= 12\times 6 = 72 > 0.$$

and $\frac{\partial^2 f}{\partial x^2} = 6x = 6\times 2 = 12 > 0$

$\frac{\partial^2 f}{\partial y^2} = 6y = 6\times 1 = 6 > 0$

Point	$A = f_{xx}$	$B = f_{xy}$	$C = f_{yy}$	$AC-B^2$	natur
(1, 2)	6>0	0	12>0	72>0	Min P
(1, −2)	6>0	0	−12<0	−72<0	S.P
(−1, 2)	−6<0	0	12>0	−72<0	S.P
(−1, −2)	−6<0	0	−12<0	72>0	Max.P

∴ The point (2, 1) is a minimum point.

At $(2,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6\times 2 = 12 > 0 \quad A>0.$$

$$\frac{\partial^2 f}{\partial y^2} = 6y = 6\times -1 = -6 < 0 \quad C<0.$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12\times(-6) - 0 = -72 < 0$$

The $(2,-1)$ is a saddle point.

At $(-2,1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6\times -2 = -12 < 0 \quad A<0$$

$$\frac{\partial^2 f}{\partial y^2} = 6y = 6\times 1 = 6 > 0. \quad C>0$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -12\times 6 - 0 = -72 < 0.$$

$\therefore$ The point $(-2,1)$ is a saddle point.

At $(-2,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6\times -2 = -12 < 0$$

$$\frac{\partial^2 f}{\partial y^2} = 6y = 6\times -1 = -6 < 0$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -12\times -6 - 0 = 72 > 0.$$

$\therefore$ The point $(-2,-1)$ is a Maximum point.

$\therefore$ The Maximum value is,

$$f(x,y) = x^3 + y^3 - 12x - 3y + 20$$

$$f(-2,-1) = (-2)^3 + (-1)^3 - 12\times -2 - 3\times -1 + 20 = -8-1+24+3+20$$

$$= 38.$$

Max. value at $(-1,-2)$ $\Big\}$ $= (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$

$= 38$

min value at $(1,2)$ $\Big\}$ $= (1)^3 + 2^3 - 3(1) - 12(2) + 20$

$= 2.$

The minimum value of $f(x,y)$ is,

$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$

$= 2.$

4. Expand $f(x,y) = x^3+y^3-3xy$ for Maximum and

⑤. minimum values.

H.W ④ $f(x,y) = x^3+y^3-3axy$. 1997 2003.

Sol : Given: $f(x,y) = x^3+y^3-3xy$

$\frac{\partial f}{\partial x} = 3x^2-3y$ | $\frac{\partial f}{\partial y} = 3y^2-3x$ | $\frac{\partial^2 f}{\partial x \partial y} = -3$

$3x^2-3ay$ | $3y^2-3ax$ | $-3a$.

$\frac{\partial^2 f}{\partial x^2} = 6x$ | $\frac{\partial^2 f}{\partial y^2} = 6y$

$6x$

To find the stationary point:

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial y}{\partial y} = 0$.

$3x^2-3y = 0$ $\qquad$ $3y^2-3x = 0$

$3x^2-3ay=0$ $\qquad$ $3y^2-3ax=0$

$x^2-ay=0$ $\qquad$ $y^2-ax=0$

$x^2-y = 0 \rightarrow ①$ $\qquad$ $y^2-x = 0 \rightarrow ②$.

$x^2=ay \rightarrow ①$ $\qquad$ $y^2=ax \rightarrow ②$

$x^2 = y \longrightarrow ①$ $\qquad$ $y^2 = x. \longrightarrow ②$.

$x = \pm y$ $\qquad$ $y = \pm x$ $\qquad$ $y^2-x=0$.

From ① $y=x^2/a$ subs in ② $\qquad$ $x^4-x=0$.

$x(x^3-1)=0$

put $x = 0 \Rightarrow y=0$ $\qquad$ $(x^2/a)^2=ax$. $\qquad$ $x=0$, $x^3-1=0$

$x = 1 \Rightarrow y=1$ $\Rightarrow x^4/a^2 = ax$ $\qquad$ $x^3=1$

$x^4 = a^3x$ $\qquad$ $x=1$

$x^4-a^3x=0 \Rightarrow x(x^3-a^3)=0$

$\therefore$ The stationary points are $(0,0)$ and $(1,1)$.

$x=0, x=a$.

$x=0 \rightarrow ①$ $y=0$

$x=a$ in ① $y=a$.

At point $(0,0)$:

$\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$

$A=0, C=0, B=-3a$

$AC-B^2=(0)(0)-(3a)^2=-9a^2<0$

$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0-9 = -9 < 0$. Points are $(0,0)$ (a,a)

$\therefore$ The point $(0,0)$ there is no extremum value.

P	A	B	C	$AC-B^2$	Nature
$(0,0)$	0	$-3a$	0	$-9a^2<0$	S.P.
(a,a)	$6a>0$	$-3a$	$6a>0$	$27a^2>0$	Min.

At point (1, 1):

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6\times 1 = 6 > 0.$$

$A = 6a$

$C = 6a$

$B = -3a$

$AC - B^2 = (6a)(6a) - 9a^2$

$= 27a^2 > 0.$

$$\frac{\partial^2 f}{\partial y^2} = 6y = 6\times 1 = 6 > 0.$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6.6 - (-3)^2$$

$$= 36 - 9 = 27 > 0.$$

∴ The point (1, 1) (a, a). is a minimum value.

∴ The minimum value is,

$f(x, y) = x^3 + y^3 - 3xy$

$f(1, 1) = 1^3 + 1^3 - 3\times 1\times 1$

$= 1+1-3 = 2-3 = -1$

$f(1, 1) = -1.$

The minimum value..

$f(x,y) = x^3+y^3-3axy$

$= a^3+a^3-3a.a.a$

$= a^3+a^3-3a^3 = 2a^3 - 3a^3$

$f(a,a) = \underline{\underline{-a^3}}.$

2. In a plane triangle. find the Maximum value of cos A cos B cos C.

Sol: W.K.T, In a plane triangle,

In a triangle sum of angles = 180°

i.e. A+B+C = π

$A + B + C = \pi$

$C = \pi - (A+B) \rightarrow$ (A)

cos(180° - θ) = -cos θ

cos 60° or π/3 = 1/2

$\therefore \cos A \cos B \cos C = \cos A \cos B \cos(\pi - (A+B))$

$= -\cos A \cos B \cos(A+B).$

Let $f(A,B) = -\cos A \cos B \cos(A+B)$

$$\therefore \frac{\partial f}{\partial A} = -\cos B \left[\cos A - \sin(A+B) + \cos(A+B) - \sin A \right]$$

$$= \cos B \left[\sin A \cos(A+B) + \cos A \sin(A+B) \right]$$

$$\frac{\partial f}{\partial A} = \cos B \sin(2A+B).$$

$\sin A \cos B + \cos A \sin B = \sin(A+B$

$\sin(A+A+B) = \sin(2A+B).$

Similarly,

$$\frac{\partial f}{\partial B} = \cos A \sin(A+2B)$$

$$\frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos(2A+B)$$

$$\frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos(A+2B)$$

$$\frac{\partial^2 f}{\partial A \partial B} = \cos B \cos(2A+B) + \sin(2A+B) - \sin B.$$

$$= \cos B \cos(2A+B) - \sin B \sin(2A+B)$$

$$= \cos(B+2A+B)$$

$\cos A \cos B - \sin A \sin B = \cos(A+B$

$\cos n\pi = (-1)^n$

$$= \cos(2A+2B)$$

$\cos \pi = (-1)^1 = -1$

$\cos 2\pi = (-1)^2 = 1.$

To find the stationary points:

$$\frac{\partial f}{\partial A} = 0 \qquad \frac{\partial f}{\partial B} = 0.$$

$\cos B \sin(2A+B) = 0 \qquad \cos A \sin(A+2B) = 0.$

ie $\cos B = 0$ (or) $\sin(2A+B) = 0$. $\cos A = 0$ or $\sin(A+2B) = 0$

ie, $\cos B = 0$

$\Rightarrow B = \cos^{-1} 0$

$\Rightarrow B = \pi/2$

(or)

$\sin(2A+B) = 0$

$2A + B = \sin^{-1} 0$

$-2A + B = 0$ (or) $\pi \to ①$.

and

$\cos A = 0$ $\quad \cos \pi/2 = 0$

$\Rightarrow A = \cos^{-1} 0$ $\quad \pi/2 = \cos^{-1} 0.$

$\Rightarrow A = \pi/2$

$\sin \pi = 0$

$\pi = \sin^{-1} 0.$

$\sin(A+2B) = 0$

$(A+2B) = \sin^{-1} 0$

$A + 2B = 0$ or $\pi \to ②.$

$\therefore A = \pi/2$ and $B = \pi/2.$

$2A + B = \pi \to ①$

$A + 2B = \pi \to ②$

$① \Rightarrow 2A + B = \pi$

$② \times 2 \Rightarrow 2A + 4B = 2\pi$

$-3B = -\pi$

$$\boxed{B = \pi/3}$$

Sub's $B = \pi/3$ in ① we ge

$2A + B = \pi$

$2A + \pi/3 = \pi$

$2A = \pi - \pi/3$

$= \frac{3\pi - \pi}{3} = 2\pi/3$

$$\boxed{A = \pi/3}$$

$\therefore$ The stationary points are ~~$(\pi/2, \pi/2)$~~, $(\pi/3, \pi/3)$.

At point $(\pi/2, \pi/2)$.

$$\frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos(2A+B) \qquad \cos \pi/2 = 0.$$

$$= 2 \cos \pi/2 \cos(2 \times \pi/2 + \pi/2)$$

$$\frac{\partial^2 f}{\partial x^2} = 0.$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \cos \pi/2 \cos(\pi/2 + 2 \times \pi/2)$$

$$= 0$$

$$\frac{\partial^2 f}{\partial A \partial B} = \cos(\cancel{2} \times \pi/\cancel{2} + \cancel{2} \times \pi/\cancel{2}) = \cos(\pi+\pi) = \cos 2\pi$$

$$= 1$$

$$\therefore \frac{\partial^2 f}{\partial A^2} \cdot \frac{\partial^2 f}{\partial B^2} - \left(\frac{\partial f}{\partial A \partial B}\right) = 0.0 - 1 = -1 < 0.$$

The point $(\pi/2, \pi/2)$ is no extremum value.

At point $(\pi/3, \pi/3)$

$$\frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos(2A+B) = 2 \cos \pi/3 \cos(2 \times \pi/3 + \pi/3)$$

$$\frac{2\pi}{3} + \pi/3 = \cancel{3}\pi/\cancel{3}$$

$$= 2 \cos \pi/3 \cos \cancel{3\pi} \pi$$

$\cos \pi/3 = 1/2$

$\cos \pi = -1$

$$= \cancel{2}\ 1/\cancel{2}\ (-1)$$

$$= -1 < 0.$$

$\cos(180+\theta)$

$= 1/2$

$\cos(4\pi/3)$

$= \cos(\pi + \pi/3)$

$= \cos(\pi/3)$

$= 1/2$

$$\frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos(A+2B) = 2 \cos \pi/3 \cos(\pi/3 + \frac{2\pi}{3})$$

$$= 2 \cos \pi/3 \cos \pi = \cancel{2} \times 1/\cancel{2} \times -1$$

$$= -1 < 0$$

$$2\pi/3 + 2\pi/3 = 4\pi/3$$

$$\frac{\partial^2 f}{\partial A \partial B} = \cos(2A+2B) = \cos(2 \times \pi/3 + 2 \times \pi/3) = \cos(4\pi/3) = \frac{-1}{2}$$

$$\therefore \frac{\partial^2 f}{\partial A^2} \cdot \frac{\partial^2 f}{\partial B^2} - \left(\frac{\partial^2 f}{\partial A \partial B}\right)^2 = (-1)(-1) - (-1/2)^2 = 1 - 1/4 = 3/4 > 0$$

$\therefore$ The point $(\pi/3, \pi/3)$ is a maximum point.

sub's in Ⓐ.

ie, $A = \pi/3$, $B = \pi/3$

$$C = \pi - (A+B) = \pi - (\pi/3 + \pi/3) = \pi - 2\pi/3 = \frac{3\pi - 2\pi}{3}$$

$$\boxed{C = \pi/3.}$$

$\therefore$ cosA cosB cosC is maximum when each of the angles is $\pi/3$.

The maximum value is, $f(A, B, C) = \cos A \cos B \cos C$.

$$f(\pi/3, \pi/3, \pi/3) = \cos \pi/3, \cos \pi/3 \cos \pi/3$$

$$= 1/2 \quad 1/2 \quad 1/2 = 1/8.$$

6. Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$. (23)

Sol:

(24)

Given: $f(x,y) = x^3y^2(1-x-y)$

$= x^3y^2 - x^4y^2 - x^3y^3$ Jan 2014

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$= x^2y^2(3-4x-3y)$$

H.W.

③ $f(x,y) = x^3y^2(12-x-y)$.

Ans: $(0,0)$ $(6,4)$.

max $= 6912$.

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= 6xy^2(1-2x-y)$$

$$\frac{\partial f}{\partial y} = x^3 2y - x^4 2y - x^3 3y^2$$

$$= x^3y[2-2x-3y)$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - x^3 6y$$

$$= x^3(2-2x-6y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 2y - 4x^3 2y - 3x^2 3y^2$$

$$= 6x^2y - 8x^3y - 9x^2y^2$$

$$= x^2y(6-8x-9y).$$

To find the stationary points.

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0$ $\qquad$ $x^3y(2-2x-3y) = 0.$

$\Rightarrow x=0, y=0$ & $3-4x-3y=0.$ $\qquad$ $\Rightarrow x=0, y=0, 2-2x-3y=0$

$\Rightarrow x=0, y=0, 4x+3y=3$ $\qquad$ $\Rightarrow x=0, y=0, 2x+3y=2.$

$$4x+3y=3 \longrightarrow ①$$

$$2x+3y=2 \longrightarrow ②$$

① − ② ⇒

$$\begin{aligned} 4x + \cancel{3y} &= 3 \\ -\,-2x - \cancel{3y} &= -2 \\ \hline 2x &= 1 \\ x &= \tfrac{1}{2} \end{aligned}$$

① ⇒ $4x + 3y = 3$

② × 2 ⇒ $4x + 6y = 4$

$$\begin{aligned} \cancel{4x} + 3y &= 3 \\ \cancel{4x} + 6y &= 4 \\ \hline -3y &= -1 \\ y &= \tfrac{1}{3}. \end{aligned}$$

∴ The stationary points are $(0,0)$, $(\frac{1}{2}, \frac{1}{3})$

At point $(0,0)$:

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2(1-2x-y) = 0 \qquad \frac{\partial^2 f}{\partial y^2} = x^3(2-2x-6y) = 0$$

P	A	B	C	$AC-B^2$	Nature
$(0,0)$	0	0	0	0	no e.v
$(\frac{1}{2},\frac{1}{3})$	$-\frac{1}{9}<0$	$-\frac{1}{12}<0$	$-\frac{1}{8}<0$	$\frac{1}{144}>0$	max.

$$\therefore \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

∴ At $(0,0)$ there is no extremum value.

At point $(\frac{1}{2}, \frac{1}{3})$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2(1-2x-y) = \overset{2}{\cancel{6}}\times\frac{1}{\cancel{2}}\times\frac{1}{\cancel{9}\,3}\left(1-\cancel{2}\times\frac{1}{\cancel{2}}-\frac{1}{3}\right)$$

$$= \frac{1}{3}\left(\cancel{1}-\cancel{1}-\frac{1}{3}\right) = \frac{1}{3}\times-\frac{1}{3} = -\frac{1}{9}<0.$$

$$\frac{\partial^2 f}{\partial y^2} = x^3(2-2x-6y) = \left(\frac{1}{2}\right)^3\left(2-\cancel{2}\times\frac{1}{\cancel{2}}-\overset{2}{\cancel{6}}\times\frac{1}{\cancel{3}}\right) = \frac{1}{8}(\cancel{2}-1-\cancel{2})$$

$$= -\frac{1}{8} < 0.$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^2y(6-8x-9y) = \frac{1}{4}\times\frac{1}{3}\left(6-\overset{4}{\cancel{8}}\times\frac{1}{\cancel{2}}-\overset{3}{\cancel{9}}\times\frac{1}{\cancel{3}}\right) = \frac{1}{12}(6-4-3)$$

$$= \frac{1}{12}\times-1 = -\frac{1}{12} < 0.$$

$$\therefore \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144}$$

$$= \frac{1}{144} > 0.$$

∴ The point is a maximum value.

$$\text{Maximum value} = f\left(\tfrac{1}{2}, \tfrac{1}{3}\right) = \left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{3}\right)^2\left(1-\tfrac{1}{2}-\tfrac{1}{3}\right)$$

$$= \left(\frac{1}{8}\right)\left(\frac{1}{9}\right)\left(\frac{6-3-2}{6}\right) = \frac{1}{8}\,\frac{1}{9}\,\frac{1}{6} = \frac{1}{432} //.$$

FUNCTIONS OF SEVERAL VARIABLES:

Partial Differentiation:

Partial Derivatives:

Let $u = f(x,y)$ be a function of two independent variables x and y. Differentiating u w.r.to 'x' keeping 'y' as a constant is ~~known~~ called the partial differential coefficient of 'u' w.r.to 'x' and it is denoted by $\frac{\partial u}{\partial x}$.

$\therefore$ $\frac{\partial u}{\partial x}$ means differentiate u w.r.to 'x' keeping 'y' constant

Ily $\frac{\partial u}{\partial y}$ means differentiate u w.r.to 'y' keeping 'x' constant

Note $\frac{\partial z}{\partial x} = p$ $\quad \frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial z}{\partial y} = q$ $\quad \frac{\partial^2 z}{\partial y^2} = t$

$\frac{\partial^2 z}{\partial x \partial y} = s.$

NOTE:

If $u = \partial(x,y)$, then

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{\partial(x+\Delta x, y) - \partial(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{\partial(x, y+\Delta y) - \partial(x,y)}{\Delta y}$$

Successive partial differentiation:

Let $u = f(x,y)$ be a function of two variables x and y. Then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ will ~~be~~ represent the first partial derivative of u w.r.to x and y.

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

NOTE:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

/ 1. If $u = \frac{y}{z} + \frac{z}{x}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z$

②

Sol: Given : $u = \frac{y}{z} + \frac{z}{x}$

$$\frac{\partial u}{\partial x} = 0 - \frac{z}{x^2} \qquad \frac{\partial u}{\partial y} = \frac{1}{z} + 0 \qquad \frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x \times \frac{-z}{x^2} + y \times \frac{1}{z} + z\left(\frac{-y}{z^2} + \frac{1}{x}\right)$$

$$= \frac{-z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$

$$= 0$$

2. If $u = \log(x^2+y^2+z^2)$, prove that,

④

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}$$

$\log x = 1/x$

Sol: Given : $u = \log(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^2+y^2+z^2)}(2x) = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{(x^2+y^2+z^2)(1) - x\cdot(2x)}{(x^2+y^2+z^2)^2}\right]$$

$$= \frac{2[x^2+y^2+z^2-2x^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2[y^2+z^2-x^2]}{(x^2+y^2+z^2)^2} \longrightarrow (1)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{2[x^2+z^2-y^2]}{(x^2+y^2+z^2)^2} \longrightarrow (2)$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{2[x^2+y^2-z^2]}{(x^2+y^2+z^2)^2} \longrightarrow (3)$$

Adding of ①, ② and ③ we get ②

$$\therefore \frac{\partial u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x^2} = \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}+\frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}+\frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{2[\cancel{y^2}+\cancel{z^2}-\cancel{x^2}+\cancel{x^2}+z^2-\cancel{y^2}+x^2+y^2-\cancel{z^2}]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2(x^2+y^2+\cancel{z^2})}{(x^2+y^2+z^2)^{\cancel{2}}}$$

$$= \frac{2}{(x^2+y^2+z^2)}$$ ✓

⑤ if $x=r\cos\theta$, $y=r\sin\theta$ find

i) $\partial x/\partial r$ ii) $\partial y/\partial \theta$ iii) $\partial r/\partial x$ iv) $\frac{\partial \theta}{\partial y}$

Sol: $x=r\cos\theta$, $y=r\sin\theta$

i) $\partial x/\partial r = \cos\theta$ ii) $\frac{\partial y}{\partial \theta} = r\cos\theta$

iii) $\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$ $r^2=x^2+y^2$

iv) $\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2}(1/x)$ $\theta = \tan^{-1} y/x$

$$= \frac{x}{x^2+y^2}.$$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{2x}{2r}$

$= x/r$

3. If $u = \sin^{-1}\left[\frac{x^3-y^3}{x+y}\right]$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\tan u$

Sol:

Given: $\sin u = \frac{x^3-y^3}{x+y}$

Partial diff. w.r.to 'x' we get,

$$\cos u\,\frac{\partial u}{\partial x} = \frac{(x+y)(3x^2)-(x^3-y^3)(1)}{(x+y)^2}$$

$$\cos u\; x.\frac{\partial u}{\partial x} = \frac{(x+y)\,3x^3-(x^3-y^3)x}{(x+y)^2}$$

$$x\frac{\partial u}{\partial x} = \frac{1}{\cos u}\left\{\frac{3x^3(x+y)-x(x^3-y^3)}{(x+y)^2}\right\} \longrightarrow ①$$

Partial diff. w.r. to 'y' we get,

$$\cos u\,\frac{\partial u}{\partial y} = \frac{(x+y)(-3y^2)-(x^3-y^3)(1)}{(x+y)^2}$$

$$y\frac{\partial u}{\partial y} = \frac{1}{\cos u}\,\frac{(-3y^3)(x+y)-y(x^3-y^3)}{(x+y)^2} \;\llcorner ②$$

Add (1)+(2)(3) we

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{\cos u}\left[\frac{3x^3(x+y) - x(x^3-y^3) - 3y^3(x+y) - y(x^3-y^3)}{(x+y)^2}\right]$$

$$= \frac{1}{\cos u}\left[\frac{3(x+y)(x^3-y^3) - (x^3-y^3)(x+y)}{(x+y)^2}\right]$$

$$= \frac{1}{\cos u} \cdot \frac{2\cancel{(x+y)}(x^3-y^3)}{(x+y)^{\cancel{2}}}$$

$$= \frac{1}{\cos u}\; 2\; \frac{x^3-y^3}{(x+y)} = 2\,\frac{1}{\cos u} \cdot \sin u$$

$$= 2\tan u.$$

4. If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

(3) (4). U.Q

Sol

Given: $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$\frac{\partial}{\partial x}\left(x/y + y/z + z/x\right) + \frac{\partial}{\partial y}(\quad) = 1/(x/y + y/z + z/x)$

$\Rightarrow \left(\frac{1}{y} - \frac{z}{x^2}\right) + \left(\frac{1}{z} - \frac{x}{y^2}\right) = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{z}{x^2} \Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{y} - \frac{xz}{x^2} = \frac{x}{y} - \frac{z}{x}$$

$\frac{1}{y} - \frac{z}{x^2} + \frac{1}{z} - \frac{x}{y^2} - \frac{x}{y} + \frac{z}{x}\; z/y = 0$

$$\frac{\partial u}{\partial y} = \frac{1}{z} - \frac{x}{y^2} \Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{z} - \frac{xy}{y^2} = \frac{y}{z} - \frac{x}{y}$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x} \Rightarrow z\frac{\partial u}{\partial z} = \frac{-y\cancel{z}}{z^{\cancel{2}}} + \frac{z}{x} = \frac{-y}{z} + \frac{z}{x}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y} - \frac{z}{x} + \frac{y}{z} - \frac{x}{y} - \frac{y}{z} + \frac{z}{x}$$

$$= 0.$$

(5)

H.W

1. if $u = (x-y)^4 + (y-z)^4 + (z-x)^4$. S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(1) MS-I 2. If $r^2 = x^2+y^2$ than S.T $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[(\partial r/\partial x)^2 + (\partial r/\partial y)^2\right]$

5. If $u = (x-y)(y-z)(z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Sol: Given: $u = (x-y)(y-z)(z-x)$ (∵ By Product rule)

①

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right] \checkmark$$

$$= -(x-y)(y-z) + (y-z)(z-x).$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial x} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$

$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -(x-y)(y-z) + (y-z)(z-x) + (x-y)(z-x) - (y-z)(z-x) + (x-y)(y-z) - (x-y)(z-x).$$

$$= 0.$$

Euler's Theorem for homogeneous functions:

(i) If u is a homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

(ii) If u is a homogeneous function of degree n in x, y and z then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \underline{\underline{nu}}$.

1. verify Euler's theorem for the function

① i) $u = x^2+y^2+2xy$ ii) $u = x^3+y^3+z^3+5xyz$.

Sol: i) Given: $u = x^2+y^2+2xy$

This is a homogeneous function of degree 2.

$$\frac{\partial u}{\partial x} = 2x+2y$$

$$x\frac{\partial u}{\partial x} = 2x^2+2xy \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = 2y+2x$$

$$y\frac{\partial u}{\partial y} = 2y^2+2xy \longrightarrow (2)$$

① + ②, weget, The Euler formula is,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2+2xy+2y^2+2xy$$

$$=2\left[x^2+y^2+xy+xy\right]$$

$$=2\left[x^2+y^2+2xy\right]$$

$$= 2u$$

Hence Euler's theorme is verified.

ii) Given: $u = x^3+y^3+z^3+3xyz$

homogeneous function of degree 3.

$$\frac{\partial u}{\partial x} = 3x^2+3yz$$

$$x\frac{\partial u}{\partial x} = 3x^3+3xyz \longrightarrow ①$$

$$\frac{\partial u}{\partial y} = 3y^2+3xz$$

$$y\frac{\partial u}{\partial y} = 3y^3+3xyz \longrightarrow ②$$

$$\frac{\partial u}{\partial z} = 3z^2+3xy$$

$$z\frac{\partial u}{\partial z} = 3z^3+3xyz \longrightarrow ③.$$

①+②+③ we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 3x^2+3xyz+3y^2+3xyz+3z^2+3xyz$$

$$= 3\left[x^2+y^2+z^2+3xyz\right]$$

$$= 3u.$$

U.Q J-2014, 2008, 2017, 2008,

2. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$

②

Sol: Given: $u = \tan^{-1}\left(\frac{x^3+y^3}{(x-y)}\right)$

$\tan u = \frac{x^3+y^3}{x-y}$ is a homogeneous function of degree 2 in x and y.

By Euler's theorem,

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\tan u\cdot\frac{1}{\sec^2 u} = \frac{2\sin u}{\cos u}\cdot\cos^2 u$$

$$= 2\sin u\cos u$$

$$= \sin 2u.$$

3. If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u.$

③ U-Q 2003, 2017

Sol: Given: $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$\frac{x^1}{x^{1/2}}$ $x^{1-1/2}$ $x^{2\cdot 1/2}$ $x^{1/2}$.

$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a H.F in x & y of degree $\frac{1}{2}$.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$$

$$x\frac{\partial}{\partial x}(\cos u)+y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$x\,-\sin u\frac{\partial u}{\partial x}+y\,-\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$

$$-\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u.$$

4. If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

④ U.Q 2009

Sol: Given: $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ is a H.F in x & y of degree 0.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$$

$$x \frac{\partial}{\partial x}(\sin u) + y \, \partial/\partial y \, (\sin u) = u(0)$$

$$x \quad \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 0.$$

$$\cos u \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$$

5. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$

ⓧ **Sol** : Given: $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

U.Q
2011, 2014,
2016, 2015.
2008,

$\sin u = \frac{x^2+y^2}{x+y}$ is a H.F of degree 1.

By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$x \, \partial/\partial x \, (\sin u) + y \, \partial/\partial y \, (\sin u) = 1 . \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \sin u$$

$$\cos u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

H.W

1. $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$

P.t $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u.$

⑦ 2. If $u = \log\left(\frac{x^3+y^3}{x-y}\right).$

P.T $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2.$

Sol $e^u = \frac{x^3+y^3}{x-y}$ degree 2.

$$x \frac{\partial e^u}{\partial x} + y \frac{\partial e^u}{\partial y} = 2e^u$$

$$e^u x \frac{\partial u}{\partial x} + e^u y \frac{\partial u}{\partial y} = 2e^u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2e^u}{e^u}$$

TAYLOR'S EXPANSION:

Def: The Taylor's series expansion for single variable is, $f(x+h) = f(x) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$

Let $f(x,y)$ be a function of too variables x, y at the point (a,b) then the taylor series expansion can be writen as,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \dots$$

where, $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$ $f_{xy} = \frac{\partial f}{\partial x \partial y}$,

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$ $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $h = x - a$

$k = y - b$

Problems:

1. (i) Expand $e^x \cos y$ about $\left(0, \frac{\pi}{2}\right)$ upto the third term using Taylor's series. (ii) $e^x \cos y$ in powers of x and y as far as the terms of the third degree.

(1) (X) 1996 2009 2010 2011 2014 2008

Sol:

Function	value at $(0, \pi/2)$
$f(x,y) = e^x \cos y$	$f(0, \pi/2) = 0.$
$f_x = e^x \cos y$	$f_x = 0$
$f_y = -e^x \sin y$	$f_y = -e^0 \sin \pi/2 = -1$
f	

$x \rightarrow a$

$y \rightarrow b.$

$f_{xx} = e^{x} \cos y$ $\qquad\qquad f_{yyy}$

$f_{xx} = e^{0} \cos \pi/2 = 0$

$f_{xy} = -e^{x} \sin y \Rightarrow f_{xy} = -e^{0} \sin \pi/2 \Rightarrow f_{xy} = -1$

$f_{yy} = -e^{x} \cos y \Rightarrow f_{yy} = -e^{0} \cos \pi/2 \Rightarrow f_{yy} = 0$

$f_{xxx} = e^{x} \cos y \Rightarrow f_{xxx} = e^{0} \cos \pi/2 \Rightarrow f_{xxx} = 0.$

$f_{xxy} = -e^{x} \sin y \Rightarrow f_{xxy} = -e^{0} \sin \pi/2 \Rightarrow f_{xxy} = -1$

$f_{xyy} = -e^{x} \cos y \Rightarrow f_{xyy} = -e^{0} \cos \pi/2 \Rightarrow f_{xyy} = 0.$

$f_{yyy} = +e^{x} \sin y \Rightarrow f_{yyy} = e^{0} \sin \pi/2 \Rightarrow f_{yyy} = 1.$

By Taylor's Theorem,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

$a = 0, \quad b = \pi/2$

$h = x - a = x$

$k = y - b = (y - \pi/2).$

$$f(x,y) = 0 + \left[x(0) + (y - \pi/2)(-1)\right] + \frac{1}{2!}\left[x^2(0) + 2x(y-\pi/2)(-1) + (y-\pi/2)^2(0)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2 \overset{(1)}{\cancel{(0)}} + (y-\pi/2)^3(0)\right] + \cdots\cdots$$

⑤

$$= -y + \frac{\pi}{2} + \frac{1}{2!}\left[-2xy + 2x\frac{\pi}{2}\right] + \frac{1}{3!}\left[-3x^2y + 3x^2\frac{\pi}{2}\right]$$

$$= -y + \frac{\pi}{2} - xy + \frac{x\pi}{2} - \frac{x^2y}{2} + \frac{x^2\pi}{4} + \cdots$$

$$= -y - xy - \frac{x^2y}{2} + \frac{\pi}{2} + \frac{x\pi}{2} + \frac{x^2\pi}{4} + \cdots$$

2. Expand $e^x \sin y$ about $(1, \pi/2)$ upto the third term using Taylor's series.

① H.W

$x \to a, y \to b$

Sol: Given that $f(x,y) = e^x \sin y$ and $(a,b) = (1, \pi/2)$

$f(x,y) = e^x \sin y$, $f(a,b) = f(1,\pi/2) = e^1 \sin \pi/2 = e$

$f_x(x,y) = e^x \sin y$, $f_x(a,b) = f_x(1, \pi/2) = e^1 \sin \pi/2 = e$

$f_{xx}(x,y) = e^x \sin y$, $f_{xx}(a,b) = f_{xx}(1, \pi/2) = e^1 \sin \pi/2 = e$

$f_{xxx}(x,y) = e^x \sin y$, $f_{xxx}(a,b) = f_{xxx}(1, \pi/2) = e^1 \sin \pi/2 = e.$

$f_y(x,y) = e^x \cos y$, $f_y(a,b) = f_y(1, \pi/2) = e^1 \cos \pi/2 = 0$

$f_{yy}(x,y) = -e^x \sin y$, $f_{yy}(a,b) = f_{yy}(1, \pi/2) = -e^1 \sin \pi/2 = -e$

$f_{yyy}(x,y) = -e^x \cos y$, $f_{yyy}(a,b) = f_{yyy}(1, \pi/2) = -e^1 \cos \pi/2 = 0.$

$f_{xy}(x,y) = e^x \cos y$, $f_{xy}(a,b) = f_{xy}(1, \pi/2) = e^1 \cos \pi/2 = 0$

$f_{xxy}(x,y) = e^x \cos y$, $f_{xxy}(a,b) = f_{xxy}(1, \pi/2) = e^1 \cos \pi/2 = 0$

$f_{xyy}(x,y) = -e^x \sin y$, $f_{xyy}(a,b) = f_{xyy}(1, \pi/2) = -e^1 \sin \pi/2 = -e$

By Taylor theorem:

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$

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$$+\frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

$a = 1 \quad , \quad b = \pi/2$

$h = x - a = x - 1$

$k = y - b = y - \pi/2 .$

$$f(x,y) = e + \left[(x-1)e + (y-\pi/2)(0)\right] + \frac{1}{2!}\left[(x-1)^2 e + 2(x-1)(y-\pi/2)0 + (y-\pi/2)^2(-e)\right] + \frac{1}{3!}\left[(x-1)^3 e + 3(x-1)^2(y-\pi/2)0 + 3(x-1)(y-\pi/2)^2(-e) + (x-1)^3(0)\right] + \cdots$$

$$= e + (x-1)e + \frac{1}{2!}\left[(x-1)^2 e - (y-\pi/2)^2 e\right] + \frac{1}{3!}\left[(x-1)^3 e - 3e(x-1)(y-\pi/2)^2\right]$$

② 7. Expand the function $\sin xy$ inpowers of $x-1$ and $y - \frac{\pi}{2}$ upto second degree terms.

1999
2014.
2015 **Sol** : Given that:

$f(x,y) = \sin xy \qquad x - 1 = 0 \qquad y - \pi/2 = 0$

$\qquad x = 1 = a \qquad y = \pi/2 = b.$

$f(x,y) = \sin xy \ , \ f(1, \pi/2) = \sin \pi/2 = 1$

$f_x = y\cos(xy) \ , \ f_x(1, \pi/2) = \pi/2 \cos \pi/2 = 0.$

$f_{xx} = -y^2 \sin(xy) \ . \ f_{xx}(1, \pi/2) = -(\pi/2)^2 \sin(\pi/2)$

$= -\frac{\pi^2}{4} .$

$\therefore f_y = x\cos(xy)$, $f_y(1,\pi/2) = x \cos\pi/2 = 0$

$f_{yy} = -x^2 \sin(xy)$, $f_{yy}(1,\pi/2) = -(1)^2 \sin\pi/2 = -1$

$$f_{xy} = y - \sin(xy)x + \cos(xy).1$$

$$= -xy\sin(xy) + \cos(xy)$$

$$f_{xy}(1,\pi/2) = -1\cdot\pi/2 \sin(\pi/2) + \cos(\pi/2)$$

$$= -\pi/2 \times 1 + 0$$

$$= -\pi/2.$$

Taylor's series expansion is,

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \ldots$$

Here $h = x-a = x-1$

$k = y-b = y-\pi/2.$

$$= 1 + [(x-1)0 + (y-\pi/2)0] + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + \cancel{2}(x-1)(y-\pi/2)(-\pi/\cancel{2}) + (y-\pi/2)^2(-1)\right]$$

$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right] + \ldots$$

2001
2009

4. Expand $f(x,y) = e^{xy}$ in Taylor series at $(1,1)$ upto second degree. ③

Sol: Given: $f(x,y) = e^{xy}$ $(a,b) = (1,1)$

$f(x,y) = e^{xy}$, $f(x,y) = e^{1\cdot1} = e$

$f_x = e^{xy}.y$ $f_x = 1.e = e.$

$f_{xx} = y^2 e^{xy}$, $f_{xx}(1,1) = e$

$f_y = x e^{xy}$, $f_y(1,1) = e$

$f_{yy} = x^2 e^{xy}$, $f_{yy}(1,1) = e$

$f_{xy} = xy e^{xy}$, $f_{xy}(1,1) = e.$

By Taylor series expansion.

$$\therefore e^{xy} = e + \frac{(x-1)e + (y-1)e}{1!} + \frac{(x-1)^2 e + 2(x-1)(y-1)e + (y-1)^2 e}{2!} + \dots$$

$$= e\left[1 + \frac{(x-1)+(y-1)}{1!} + \frac{(x-1)^2 + 2(x-1)(y-1) + (y-1)^2}{2!} + \dots\right]$$

5. Expand $e^x \log(1+y)$ in powers of x and y upto

[illegible] terms of third degree.

$(a,b) = (0,0)$

2010
2014
2016.

Sol: Given: $f(x,y) = e^x \log(1+y)$ $f(0,0) = 0.$

$f_x = e^x \log(1+y)$, $f_x(0,0) = e^0 \log(1+0) = 0$

$f_{xx} = e^x \log(1+y)$, $f_{xx}(0,0) = 0$

$f_{xxx} = e^x \log(1+y)$, $f_{xxx}(0,0) = 0.$

$f_y = e^x \frac{1}{1+y} = e^x (1+y)^{-1}$. $f_y(0,0) = e^0 (1+0)^{-1} = 1$

$f_{yy} = -e^x (1+y)^{-2}$, $f_{yy}(0,0) = -e^0(1+0)^{-2} = -1$

$f_{yyy} = 2e^x (1+y)^{-3}$, $f_{yyy}(0,0) = 2e^0(1+0)^{-3} = 2$

$f_{xy} = e^x (1+y)^{-1}$ $f_{xy}(0,0) = e^0(1+0)^{-1} = 1$

$f_{xxy} = e^x (1+y)^{-1}$ $f_{xxy}(0,0) = e^0(1+0)^{-1} = 1$

$\therefore f_{xyy} = -e^{x}(1+y)^{-2}, \quad f_{xyy}(0,0) = -e^{0}(1+0)^{-2} = -1.$

$$e^{x}\log(1+y) = 0 + \frac{x(0)+y(1)}{1!} + \frac{x^2(0)+2xy(1)+y^2(-1)}{2!} + \frac{x^3(0)+3x^2y(1)+3xy^2(-1)+y^3(2)}{3!} + \cdots$$

$$= \frac{y}{1!} + \frac{2xy-y^2}{2!} + \frac{3x^2y-3xy^2+2y^3}{3!} + \cdots$$

~~①~~ $\tan^{-1}(y/x)$.

Find the Taylor's Series expansion of $x^2y^2+2x^2y+3xy^2$ in powers of $(x+2)$ and $(y-1)$ upto 3rd degree terms.

H.W

② Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ upto the third degree terms.

TOTAL Derivatives: Differentiation of implict functions:

If $u = f(x, y)$ is a function of x and y,

where $x = f(t)$ and $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \longrightarrow ①$$

① can be written as,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

du is called the total differential of u.

NOTE:

1. $\frac{du}{dx} = \frac{\partial u}{\partial x} \frac{dx}{dx} + \frac{\partial u}{\partial y} \frac{dy}{dx}$

$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$

2. $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$

3. Differentiation of implict funt-

Let $f(x, y) = c$,

then $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$.

$\left[\because \frac{\partial f}{\partial y} \neq 0\right]$.

1. Find $\frac{du}{dt}$ if $u = x^3 y^4$ where $x = t^3$ and $y = t^2$.

① Sol: W.k.T $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

$u = x^3 y^4$ $\qquad x = t^3$

$\frac{\partial u}{\partial x} = 3x^2 y^4$ $\qquad \frac{dx}{dt} = 3t^2$

$y = t^2$

$\frac{\partial u}{\partial y} = x^3 4y^3$ $\qquad \frac{dy}{dt} = 2t$

$$\therefore \frac{du}{dt} = 3x^2y^4 . 3t^2 + 4x^3y^3 . 2t$$

$$= 3(t^3)^2 (t^2)^4 3t^2 + 4(t^3)^3 (t^2)^3 . 2t$$

$$= 3t^6 t^8 3t^2 + 4t^9 t^6 2t$$

$$= 9t^{16} + 8t^{16}$$

$$\frac{du}{dt} = 17t^{16}.$$

H.W 1. if $u = xy + yz + zx$, where $x = 1/t$, $y = e^t$, $z = e^{-t}$, find $\frac{du}{dt}$

2. if $u = x^2 y^3$, $x = \log t$, $y = e^t$. find $\frac{du}{dt}$

② / 2. Find $\frac{dy}{dx}$ when $x^3+y^3 = 3axy$.

Sol: Let $f(x,y) = x^3+y^3-3axy$. **Ans**

$$\frac{\partial f}{\partial x} = 3x^2-3ay$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{\cancel{3}x^2-\cancel{3}ay}{\cancel{3}y^2-\cancel{3}ax} = -\frac{x^2-ay}{y^2-ax}$$

$$\frac{dy}{dx} = -\frac{x^2-ay}{y^2-ax}.$$

3. Find y' when $y\sin x = x\cos y$

$f(x,y) = y\sin x - x\cos y$

$\frac{\partial f}{\partial x} = y\cos x - \cos y$

$\frac{\partial f}{\partial y} = \sin x + x\sin y$

$\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{\partial f/\partial y} = -\left(\frac{y\cos x - \cos y}{\sin x + x\sin y}\right)$

Jan 2014 2015 / 2016. 4. if $x^y+y^x=c$ find dy/dx

$$\frac{d(a^x)}{dx} = a^x\log a$$

Sol Let $f(x,y) = x^y+y^x-c=0$

$\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

⑦ 3. If $u = x^2+y^2+z^2$ and $x = e^t$, $y = e^t\sin t$, $z = e^t\cos t$,

Find $\frac{du}{dt}$.

/ NOV-2001.

Sol: W.K.T $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

Given: $u = x^2+y^2+z^2$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial z} = 2z$

$x = e^t$ $\quad y = e^t\sin t$

$\frac{dx}{dt} = e^t$ $\quad \frac{dy}{dt} = (e^t\cos t + \sin t\, e^t)$

$z = e^t\cos t$

$\frac{dz}{dt} = -e^t\sin t + e^t\cos t.$

$$\frac{du}{dt} = 2x\,e^t + 2y(e^t\sin t + e^t\cos t) + 2z(e^t\cos t - e^t\sin t)$$

$$= 2e^t\left[x + y(\sin t+\cos t) + z(\cos t - \sin t)\right]$$

5. If $(\cos x)^y = (\sin y)^x$ find dy/dx.

Sol Taking log on b.s

$y\log(\cos x) = x\log(\sin y)$

$f(x,y) = y\log(\cos x) - x\log(\sin y) = 0$

$f_x = -y\tan x - \log\sin y$

$f_y = \log\cos x - x\cot y.$

$\therefore dy/dx = -f_x/f_y = \frac{y\tan x+\log\sin y}{\log\cos x - x\cot y}.$

⑦

$$= 2e^t \left[e^t + e^t \sin t\,(\sin t + \cos t) + e^t \cos t\,(\cos t - \sin t)\right]$$

$$= 2e^t \left[e^t + e^t \sin^2 t + \cancel{e^t \sin t \cos t} + e^t \cos^2 t - \cancel{e^t \sin t \cos t}\right]$$

$$= 2e^t \left[e^t + e^t (\sin^2 t + \cos^2 t)\right]$$

$$= 2e^t \left[e^t + e^t\right] = 2e^t \; 2e^t = 4e^{2t}$$

⑧ U.Q 1997 4. If $u = x \log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

(✗) 4.Q 2003, 2011, 2012.

2002 2009

Sol:

$$u = x \log(xy)$$

$$= x(\log x + \log y)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= x\left[\frac{1}{x}\right] + (\log x + \log y) + x\left(\frac{1}{y}\right)\cdot\frac{dy}{dx}$$

2015.

$$= 1 + (\log x + \log y) + \frac{x}{y}\frac{dy}{dx} \longrightarrow ①.$$

Given: $x^3 + y^3 + 3xy = 1$

Diff w.r.to x,

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(y + x\frac{dy}{dx}\right) = 0$$

$$x^2 + y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$(x^2 + y) + (x + y^2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(y + x^2)}{x + y^2}$$

$$(1) \Rightarrow \quad \frac{du}{dx} = 1 + \log x + \log y - \frac{x}{y}\frac{(y + x^2)}{(x + y^2)}$$

H.W 5) If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$, $v = 2xy$ then

P.T $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2 + y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$

(✗) 2013, 2014, 2016, 2008,

6. If $z = f(y-z, z-x, x-y)$

S.T $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} = 0.$

Sol Let $u = y - z$, $v = z - x$

$w = x - y$.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\,\partial w/\partial x.$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1)$$

$$\frac{\partial z}{\partial x} = -\frac{\partial f}{\partial v} + \partial f/\partial w$$

$$\text{Ily}\quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial z}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

Adding, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0$

5. If $z = u(x,y)$ where $x = e^{u}\cos v$ and $y = e^{u}\sin v$

I.Q ⑦ 2007 2000 Show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.

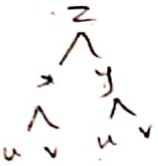
Sol :

Given : $z = u(x, y)$ $x = e^{u}\cos v$ $y = e^{u}\sin v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

z = u(x,y).

$$= \frac{\partial z}{\partial x}\, e^{u}\cos v + \frac{\partial z}{\partial y}\, e^{u}\sin v$$



$$\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \rightarrow ①$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{u}\sin v) + \frac{\partial z}{\partial y}(e^{u}\cos v)$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}\, y + \frac{\partial z}{\partial y}\, x \rightarrow ②$$

$$\therefore y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = y\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) + x\left(-\frac{\partial z}{\partial x}y + \frac{\partial z}{\partial y}x\right)$$

$$= xy\cancel{\frac{\partial z}{\partial x}} + y^2\frac{\partial z}{\partial y} - xy\cancel{\frac{\partial z}{\partial x}} + x^2\frac{\partial z}{\partial y}$$

$$= (x^2+y^2)\frac{\partial z}{\partial y}$$

$$= (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}(\sin^2 v + \cos^2 v)\, \partial z/\partial y$$

$$= e^{2u}\frac{\partial z}{\partial y}.$$

④ H.W Given the transformation $u = e^{x}\cos y$ & $v = e^{x}\sin y$ and that ϕ is a funct- of u & v and also of x & y P.T.

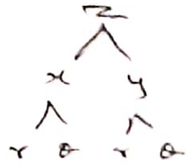
2002 2010 ② Arts

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right).$$

6. If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ S.T ⑧

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

⑩ **Sol:** Given, $x = r\cos\theta$ $y = r\sin\theta$.



Jan 2006
2014
2010
2008.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$= -r\frac{\partial z}{\partial x}\sin\theta + r\frac{\partial z}{\partial y}\cos\theta$$

$$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta.$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial^2 z}{\partial x}\right)^2\left[\cos^2\theta + \sin^2\theta\right] + \left(\frac{\partial z}{\partial y}\right)^2 \left[\sin^2\theta + \cos^2\theta\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

⑪ 7. If z is a function of x and y where

U.Q. $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$, S.T $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

2014 Sol: Given: z is a composite function of u and v.

Jan 2010

2007

2001

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^{u} + \frac{\partial z}{\partial y} (-e^{-u})$$

$$\frac{\partial z}{\partial u} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \rightarrow ①$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \partial z/\partial x\,(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y} \longrightarrow ②$$

① - ② ⇒
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial x}(e^{u} + e^{-v}) + \frac{\partial z}{\partial y}(-e^{-u} + e^{v})$$

$$= \frac{\partial z}{\partial x} \cdot x + \frac{\partial z}{\partial y} - (e^{-u} - e^{v})$$

$$= \frac{\partial z}{\partial x}x - y\frac{\partial z}{\partial y}.$$

2003
2007
2010

Abs

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

(x) If z is a function of x and y and u & v are two other

(7) variable S.T $u = lx + my$; $v = ly - mx$, S.T $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right.$

⑫ 8: If $u = \log(x^3+y^3+z^3-3xyz)$, Show that, ①

$$\left(\frac{\partial}{\partial x}+\partial/\partial y+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

U.Q.
2009
2010
2011.

Sol:

Given : $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2-3yz)$$

$$= \frac{3(x^2-yz)}{(x^3+y^3+z^3-3xyz)}$$

Illy,
$$\frac{\partial u}{\partial y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz} \quad \text{and}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2-yz+y^2-zx+z^2-xy)}{\underset{x.x^2+y.y^2+z.z^2-xyz-xyz-xyz}{x^3+y^3+z^3-3xyz}}$$

$$= \frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}$$

i.e., $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{(x+y+z)}$$

Again Diff. both the sides partially w.r. to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{-3}{(x+y+z)^2}$$

|||ly, $\frac{\partial}{\partial y}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u\right\} = \frac{-3}{(x+y+z)^2}$

$\frac{\partial}{\partial z}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u\right\} = \frac{-3}{(x+y+z)^2}$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{-9}{(x+y+z)^2}$$

k, $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

⑥ H.W - 9 U.Q. 1999

If $u = f(x,y)$ where $x = e^r\cos\theta$, $y = e^r\sin\theta$

S.T $\left(\frac{\partial u}{\partial \theta}\right)^2+\left(\frac{\partial u}{\partial r}\right)^2 = e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$ or

$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$

Sol: Given: $x = e^r\cos\theta \quad y = e^r\sin\theta$

w.k.T

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}(e^r\cos\theta) + \frac{\partial u}{\partial y}(e^r\sin\theta) \longrightarrow ①$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-e^r\sin\theta) + \frac{\partial u}{\partial y}(e^r\cos\theta) \rightarrow ②$$

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial^2 u}{\partial x^2}\right)^2 e^{2r}\cos^2\theta + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 e^{2r}\sin^2\theta$$

$$+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^r\cdot e^r\cos\theta\sin\theta + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 e^{2r}\sin^2\theta$$

$$+\left(\frac{\partial^2 u}{\partial y^2}\right)^2 e^{2r}\cos^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^r e^r\cos\theta\sin\theta.$$

$$= \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]e^{2r}$$

FUNCTIONS OF SEVERAL VARIABLES

SYLLABUS

1. PARTIAL DIFFERENTIATION
2. EULER’S THEOREM
3. TOTAL DERIVATIVES
4. CHANGE OF VARIABLES
5. JACOBIANS
6. TAYLOR’S SERIES FOR FUNCTIONS FOR TWO VARIABLES
7. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES
8. LAGRANGE’S METHOD OF UNDETERMINED MULTIPLIERS

Partial derivatives :

If $u = f(x, y, z)$, the derivative of u w.r.t x treating y and z as constants is called the partial derivative of u w.r.t x and is denoted by $\frac{\partial u}{\partial x}$ or u_x . Similarly $\frac{\partial u}{\partial y}$ is the derivative of u w.r.t y treating the other variables x and z as constants. $\frac{\partial u}{\partial z}$ is obtained by differentiating u w.r.t z treating x and y as constants.

PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function, then

(i) First order partial derivatives : $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

(ii) Second order partial derivatives : $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$

(iii) Third order partial derivatives : $\frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial y^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}$

Note:

1. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = u_{xx}$

2. $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = u_{yy}$

3. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = u_{xy}$

4. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Problems

1. **If** $u=(x-y)(y-z)(z-x)$ **show that** $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Solution:

Given $u=(x-y)(y-z)(z-x)$ then

$$\frac{\partial u}{\partial x}=(y-z)\left[(x-y)(-1)+(z-x)(1)\right]=(y-z)(z-x)-(y-z)(x-y)---(1)$$

$$\frac{\partial u}{\partial y}=(z-x)\left[(x-y)(1)+(y-z)(-1)\right]=(x-y)(z-x)-(y-z)(z-x)---(2)$$

$$\frac{\partial u}{\partial z}=(x-y)\left[(y-z)(1)+(z-x)(-1)\right]=(x-y)(y-z)-(x-y)(z-x)---(3)$$

Adding (1),(2) and (3) we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

2. **If** $u = x^y$ **then find** $(i)\ u_{xy}$ $(ii)\ u_{xyx}$

Solution

Given $u = x^y$ ---------(1)

then

(i) Differentiating (1) w.r.t 'y', we get

$$u_y = x^y \log x$$

Again differentiating w.r.t 'x' we get

$$u_{xy} = yx^{y-1}\left[\log x\right] + x^{y-1} = x^{y-1}\left(1 + y\log x\right)$$

(ii) Differentiating (1) w.r.t ' x', we get

$$u_x = yx^{y-1}$$

Again differentiating w.r.t 'y' we get

$$u_{yx} = yx^{y-1} \log x + x^{y-1}$$

Again differentiating w.r.t ‘x’ we get

$$u_{xyx} = x^{y-1}\left(\frac{y}{x}\right) + (1 + y\log x)(y-1)x^{y-2} \quad = yx^{y-2} + (1 + y\log_e x)(y-1)x^{y-2}$$

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3. **If** $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ **prove that** $z_{xy} = \frac{x^2 - y^2}{x^2 + y^2}$

Solution

Given $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

Differentiating w.r.t ‘x’ we get

$$z_x = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2\frac{1}{1 + \left(\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right) - y^2\frac{1}{1 + \left(\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) + \frac{-x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Again differentiating w.r.t ‘y’ we get

$$z_{yx} = z_{xy} = 2x\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

4. **If** $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then prove that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ **(ii)** $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 = -\frac{9}{(x+y+z)^2}$

Solution

Given $$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Then
$$\frac{\partial u}{\partial x} = \frac{3(x^2 - zy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

(i) $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}=\frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\frac{3}{x+y+z} \quad -----------------(1)$$

(ii) Operating $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)$ on both sides of (1), we get

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$

5. **If $\mathbf{x = r\cos\theta, y = r\sin\theta}$** prove that

(i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \quad (x\neq 0, y\neq 0)$

Solution:

$x = r\cos\theta, y = r\sin\theta.$

$\therefore x^2 + y^2 = r^2$ and $\tan\theta = y/x$

Differentiating $r^2 = x^2 + y^2$ partially w.r.t x, we get

$$2r.\frac{\partial r}{\partial x} = 2x \qquad \text{i.e.,} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ ------------------------------(1)}$$

Differentiating $r^2 = x^2 + y^2$ partially w.r.t y, we get

$$2r.\frac{\partial r}{\partial y} = 2y \qquad \text{i.e.,} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \text{ ---------------------(2)}$$

$$\therefore \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]$$

$$=\frac{1}{r}.\frac{1}{r^2}(x^2+y^2)$$

$$=\frac{1}{r} \text{ ----------------------(3)}$$

Differentiating (1) partially w.r.t x, we get

$$\frac{\partial^2 r}{\partial x^2}=x\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}+1.\frac{1}{r}$$

$$=\left(\frac{-x}{r^2}\right).\frac{x}{r}+\frac{1}{r}$$

Similarly from (2), we get,

$$\frac{\partial^2 r}{\partial y^2} = y\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + 1.\frac{1}{r}$$

$$= \left(\frac{-y}{r^2}\right).\frac{y}{r} + \frac{1}{r}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = -\frac{1}{r^3}(x^2 + y^2) + \frac{2}{r}$$

$$= -\frac{1}{r} + \frac{2}{r}$$

$$= \frac{1}{r} \text{ ---------- } (4)$$

From (3) and (4), we get $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$

Homogeneous function :

A function f(x,y) is said to be a homogeneous function of degree n in x and y if

$$f(x,y) = x^n F\left(\frac{y}{x}\right) \quad \text{or} \quad f(x,y) = y^n G\left(\frac{x}{y}\right)$$

Euler's Theorem

If f(x, y) is a homogenous function of degree n in x and y, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

If f(x, y) is a homogenous function of degree n in x and y, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n\,(n-1)\,f.$$

Problems:

1. **If u is a homogeneous function of degree n in x and y, show that**

$$(i)\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$(ii)\ Given\ u(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right).$$

Find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

Solution:

(i) By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \ldots\ldots\ldots..(1)$

Differentiating (1) partially w.r.to x,we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

$$(i.e)x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}\ldots\ldots. (2)$$

Differentiating (1) partially w.r.to y,we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

$$(i.e)y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial y\partial x}=(n-1)\frac{\partial u}{\partial y}\quad ....(3)$$

$$(2)\times x+(3)\times y\Rightarrow$$

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial y\partial x}$$

$$=(n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}$$

$$=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=(n-1)(nu)\text{ by }(1)$$

But $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

(ii) $u(x,y)$ is a homogeneous function of degree 2.

Hence by Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=2(1)u=2u.$$

2. **If** $\boldsymbol{u=cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)}$ **prove that** $\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u}$

Solution:

Let $$\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f(x,y)$$

$$\cos u=\frac{x}{\sqrt{x}}\left(\frac{1+y/x}{1+\sqrt{y}/\sqrt{x}}\right)$$
$$=x^{1/2}F(y/x)$$

This is a homogenous function of degree $\frac{1}{2}$

By Euler's theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

$$x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=\frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x}+y(-\sin u)\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

3. **Show that** $x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}+z.\frac{\partial u}{\partial z}=2\tan u$ where $u=\sin^{-1}\left[\frac{x^3+y^3+z^3}{ax+by+cz}\right]$

Solution:

We have, $\sin u=\left[\frac{x^3+y^3+z^3}{ax+by+cz}\right]$

Let
$$f(x,y,z)=\frac{x^3+y^3+z^3}{ax+by+cz}$$

$$f(tx,ty,tz)=\frac{t^3x^3+t^3y^3+t^3z^3}{atx+bty+ctz}=t^2f(x,y,z)$$

$\therefore$ f(x, y, z) is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,

$$x.\frac{\partial f}{\partial x}+y.\frac{\partial f}{\partial y}+z.\frac{\partial f}{\partial z}=2.f$$

From (1), we have, $f = \sin u$

$$\therefore \quad \frac{\partial f}{\partial x} = \cos u.\frac{\partial u}{\partial x} \qquad \frac{\partial f}{\partial y} = \cos u.\frac{\partial u}{\partial y}$$

and

$$\frac{\partial f}{\partial z} = \cos u.\frac{\partial u}{\partial z}$$

Substituting these in (2), we get,

$$x.\cos u.\frac{\partial u}{\partial x} + y.\cos u.\frac{\partial u}{\partial y} + z.\cos u.\frac{\partial u}{\partial z} = 2.\sin u$$

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z..\frac{\partial u}{\partial z} = 2.\tan u$$

4. **If** $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$,**prove that** $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 2\sin u\cos 3u$

Solution:

Given that $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, implies

$\tan u = f(x, y) = \frac{x^3+y^3}{x-y}$ a homogenous function of degree 2.

Therefore, by Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 2f$

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\frac{\sin u}{\cos u}\cos^2 u = \sin 2u \text{-----------------------} (1)$$

Differentiating (1) partially with respect to x and multiply with x, we get,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=2x\cos 2u\frac{\partial u}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=\left(2\cos 2u-1\right)x\frac{\partial u}{\partial x}\text{-------------------(2)}$$

Differentiating (1) partially with respect to y and multiply with y, we get,

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+x\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial y}$$

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}=2y\cos 2u\frac{\partial u}{\partial y}$$

$$y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial x\partial y}=\left(2\cos 2u-1\right)y\frac{\partial u}{\partial y}\text{---------------(3)}$$

Adding (2) and (3), we get

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= (2\cos 2u - 1)\ \sin 2u\ = 2\sin u\left[4\cos^3 u - 3\cos u\right]\ = 2\sin u\ \cos 3u$$

TOTAL DERIVATIVES - CHANGE OF VARIABLES

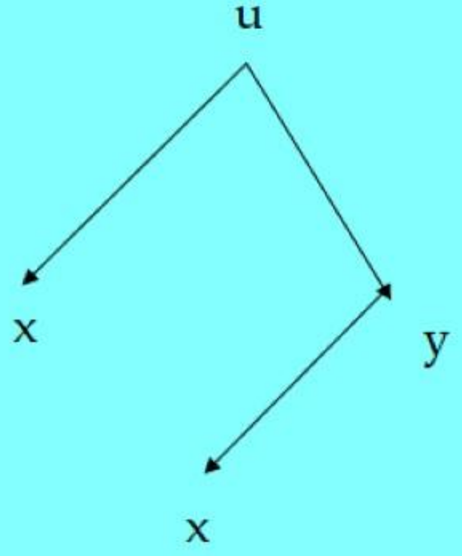
Total Derivatives:

If u is a function of two variables x and y i.e., u=u(x,y) , then the derivative $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ is called the total derivative of u.

Note:

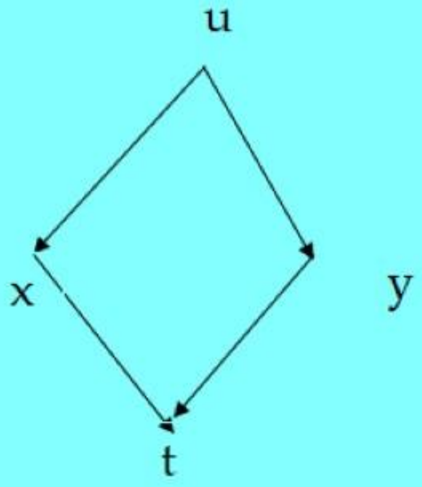
1. If u=u(x,y) and y is a function of x , then

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x}\,(1) + \frac{\partial u}{\partial y}\frac{dy}{dx}$$



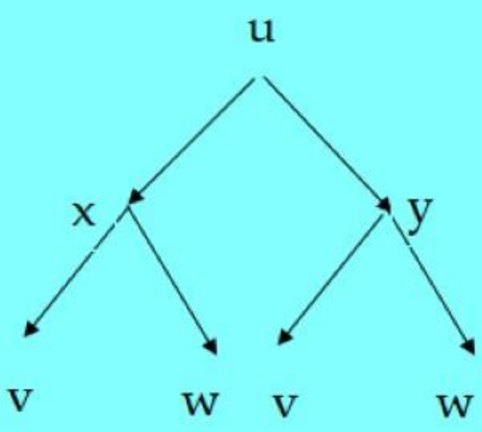
2. If u=u(x,y) and both x and y is a function of t , then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$



3. If u=u(x,y) where both x and y are function of v and w , then

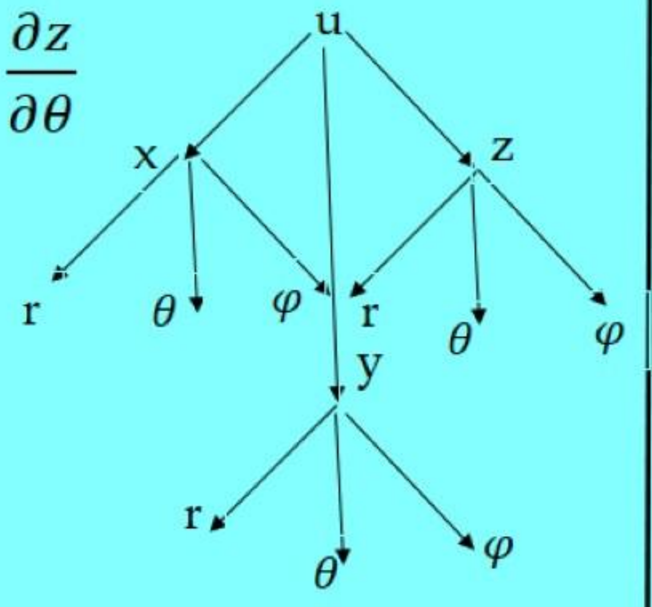
$$\frac{\partial u}{\partial v}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \quad and \quad \frac{\partial u}{\partial w}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial w}$$



4. If u=u(x,y,z) where both x, y, z are function of three other variables r, θ, φ, then

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}, \quad \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi}$$



Differentiation of Implicit Functions

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Problems:

1. **If $z = x^2 + y^2$ where $x = t^3$, $y = t^2$, find $\frac{dz}{dt}$**

Solution:

Given :

$z = x^2 + y^2$	$x = t^3$	$y = t^2$
$\frac{\partial z}{\partial x} = 2x$	$\frac{dx}{dt} = 3t^2$	$\frac{dy}{dt} = 2t$
$\frac{\partial z}{\partial y} = 2y$		

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= 2x(3t^2) + 2y(2t)$$

$$= 2t^3(3t^2) + 2t^2(2t)$$

$$= 2t^3(2 + 3t^2)$$

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2. If $z = x^2 - 3xy^2$ where $x = e^t$, $y = e^{-t}$, find $\frac{dz}{dt}$

Solution:

Given :

$z = x^2 - 3xy^2$	$x = e^t$	$y = e^{-t}$
$\frac{\partial z}{\partial x} = 2x - 3y^2$	$\frac{dx}{dt} = e^t$	$\frac{dy}{dt} = -e^{-t}$
$\frac{\partial z}{\partial y} = -6xy$		

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x - 3y^2)(e^t) - 6xy(-e^{-t})$$

$$= (2e^t - 3(e^{-t})^2)(e^t) - 6e^t e^{-t}(-e^{-t})$$

$$= (2e^{2t} - 3e^{-t}) + 6e^{-t}$$

$$= 2e^{2t} + 3e^{-t}$$

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3. If $u = x^2 + y^2 + z^2$ where $x = t\,, y = \cos t\,, z = \sin t$ find $\frac{du}{dt}$

Solution:

Given :

$u = x^2 + y^2 + z^2$	$x = t$	$y = \cos t$	$z = \sin t$
$\frac{\partial u}{\partial x} = 2x$	$\frac{dx}{dt} = 1$	$\frac{dy}{dt} = -\sin t$	$\frac{dz}{dt} = \cos t$
$\frac{\partial u}{\partial y} = 2y$			
$\frac{\partial u}{\partial z} = 2z$			

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2x)(1) + (2y)(-\sin t) + (2z)(\cos t)$$

$$= (2t)(1) + (2\cos t)(-\sin t) + (2\sin t)(\cos t)$$

$$= 2t$$

4. If $u = log(x + y + z)$ where $x = e^{-t}$, $y = cos\, t$, $z = sin\, t$, find $\frac{du}{dt}$

Solution:

Given :

$\boldsymbol{u = log(x+y+z)}$	$\boldsymbol{x = e^{-t}}$	$\boldsymbol{y = cos\ t}$	$\boldsymbol{z = sin\ t}$
$\frac{\partial u}{\partial x} = \frac{1}{x+y+z}$	$\frac{dx}{dt} = -\boldsymbol{e^{-t}}$	$\frac{dy}{dt} = -\sin t$	$\frac{dz}{dt} = \cos t$
$\frac{\partial u}{\partial y} = \frac{1}{x+y+z}$			
$\frac{\partial u}{\partial z} = \frac{1}{x+y+z}$			

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{1}{x+y+z}\right)(-\boldsymbol{e^{-t}}) + \left(\frac{1}{x+y+z}\right)(-\sin t) + \left(\frac{1}{x+y+z}\right)(\cos t)$$

$$= \left(\frac{1}{x+y+z}\right)[(-e^{-t}) + (-\sin t) + (\cos t)]$$

$$= \left(\frac{-e^{-t} - \sin t + \cos t}{e^{-t} + \cos\ t + \sin t}\right)$$

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5. If $u = xy + yz + zx$ where $x = t$, $y = e^t$, $z = t^2$, find $\frac{du}{dt}$

Solution:

Given :

$u = xy + yz + zx$	$x = t$	$y = e^t$	$z = t^2$
$\frac{\partial u}{\partial x} = y + z$	$\frac{dx}{dt} = 1$	$\frac{dy}{dt} = e^t$	$\frac{dz}{dt} = 2t$
$\frac{\partial u}{\partial y} = x + z$			
$\frac{\partial u}{\partial z} = x + y$			

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y+z)(\mathbf{1}) + (x+z)(\,\boldsymbol{e^t}) + (\boldsymbol{x}+\boldsymbol{y})(\mathbf{2}\boldsymbol{t})$$

$$= \left(\boldsymbol{e^t} + \boldsymbol{t^2}\right)(\mathbf{1}) + \left(t + \boldsymbol{t^2}\right)(\,\boldsymbol{e^t}) + (\boldsymbol{t} + \boldsymbol{e^t})(\mathbf{2}\boldsymbol{t})$$

$$= 3t^2 + e^t(1 + 3t + t^2)$$

6. If $u = x^2 + y^2 + z^2$ where $x = e^{2t}\,, y = e^{2t}\cos 3t\,,\ z = e^{2t}\sin 3t$ find $\frac{du}{dt}$

Solution:

Given :

$u = x^2 + y^2 + z^2$	$x = e^{2t}$	$y = e^{2t}\cos 3t$	$z = e^{2t}\sin 3t$
$\frac{\partial u}{\partial x} = 2x$	$\frac{dx}{dt} = 2e^{2t}$	$\frac{dy}{dt} = 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$	$\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t$
$\frac{\partial u}{\partial y} = 2y$			
$\frac{\partial u}{\partial z} = 2z$			

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2x)(2e^{2t})$$
$$+ (2y)(2e^{2t}cos\ 3t - 3e^{2t}sin\ 3t) + (2z)(2e^{2t}sin\ 3t + 3e^{2t}cos\ 3t)$$

$$= (2e^{2t})(2e^{2t})$$
$$+ (2e^{2t}cos\ 3t)(2e^{2t}cos\ 3t - 3e^{2t}sin\ 3t) + (2e^{2t}sin\ 3t)(2e^{2t}sin\ 3t$$
$$+ 3e^{2t}cos\ 3t)$$

$$= (2e^{2t})(2e^{2t})$$
$$+ (2e^{2t}cos\ 3t)e^{2t}(2cos\ 3t - 3sin\ 3t) + (2e^{2t}sin\ 3t)e^{2t}(2sin\ 3t + 3cos\ 3t)$$

$$= 4e^{4t}[1 + cos^2 3t + sin^2 3t]$$
$$= 4e^{4t}(1 + 1)$$

$$= 8e^{4t}$$

7. Find $\frac{du}{dx}$ given that $u = sin(x^2 + y^2)$ where $x^2 + y^2 = a^2$

Solution:

Given $u = sin(x^2 + y^2)$ where $x^2 + y^2 = a^2$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \text{ ------------(1)}$$

$u = sin(x^2 + y^2)$	$x^2 + y^2 = a^2$
$\frac{\partial u}{\partial x} = 2x\, cos(\boldsymbol{x^2 + y^2})$ $\frac{\partial u}{\partial y} = 2y\, cos(\boldsymbol{x^2 + y^2})$	$2x + 2y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{x}{y}$

Sub. in eqn (1)

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= 2x\, cos(x^2 + y^2) + 2y\, cos(x^2 + y^2)\left(-\frac{x}{y}\right)$$

$$= 2x\, cos(x^2 + y^2) - 2x\, cos(x^2 + y^2)$$

$$= 0$$

8. Find $\frac{dy}{dx}$ given that x cos y + y sin x = 1

Solution:

Let z=f(x,y)= x cos y + y sin x – 1

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{-------------}(1)$$

$$\frac{\partial f}{\partial x} = \cos y + y \cos x$$

$$\frac{\partial f}{\partial y} = -x \sin y + \sin x$$

Sub. in eqn (1)

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(\cos y+y \cos x)}{-x \sin y+\sin x} = \frac{\cos y+y \cos x}{x \sin y-\sin x}$$

9. **If** $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution :

Given

$$u = f(x-y, y-z, z-x) = f(r,s,t) \ where \quad r = x-y;\ s = y-z;\ t = z-x$$

$$\frac{\partial r}{\partial x} = 1, \frac{\partial r}{\partial y} = -1, \frac{\partial r}{\partial z} = 0$$

$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 1, \frac{\partial s}{\partial z} = -1$$

$$\frac{\partial t}{\partial x} = -1, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 1$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$=\ \frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$=\ \frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0$$

10. **If** $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Solution :

Given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) = f(r, s, t)\ where r = \frac{x}{y};\ s = \frac{y}{z}; t = \frac{z}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{y}, \frac{\partial r}{\partial y} = -\frac{x}{y^2}, \frac{\partial r}{\partial z} = 0$$

$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = \frac{1}{z}, \frac{\partial s}{\partial z} = -\frac{y}{z^2}$$

$$\frac{\partial t}{\partial x} = -\frac{z}{x^2}, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = \frac{1}{x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)-\frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{z}\right)+\frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{z}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

Now

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\left(\frac{x}{y}\right)+\frac{\partial u}{\partial t}\left(-\frac{z}{x}\right)+\frac{\partial u}{\partial r}\left(-\frac{x}{y}\right)+\frac{\partial u}{\partial s}\left(\frac{y}{z}\right)+\frac{\partial u}{\partial s}\left(-\frac{y}{z}\right)+\frac{\partial u}{\partial t}\left(\frac{z}{x}\right)=0$$

11. **If** $u=f\left(\frac{x-y}{xy},\frac{y-z}{yz},\frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$

Solution :

Given

$$u=f\left(\frac{x-y}{xy},\frac{y-z}{yz},\frac{z-x}{xz}\right)=f(r,s,t)where$$

$$r=\frac{x-y}{xy}=\frac{1}{y}-\frac{1}{x};\ s=\frac{y-z}{yz}=\frac{1}{z}-\frac{1}{y};\ \ t=\frac{z-x}{xz}=\frac{1}{x}-\frac{1}{z}$$

$$\frac{\partial r}{\partial x}=\frac{1}{x^2},\frac{\partial r}{\partial y}=-\frac{1}{y^2},\frac{\partial r}{\partial z}=0$$

$$\frac{\partial s}{\partial x}=0,\frac{\partial s}{\partial y}=\frac{1}{y^2},\frac{\partial s}{\partial z}=-\frac{1}{z^2}$$

$$\frac{\partial t}{\partial x}=-\frac{1}{x^2},\frac{\partial t}{\partial y}=0,\frac{\partial t}{\partial z}=\frac{1}{z^z}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$=\ \frac{\partial u}{\partial r}\left(\frac{1}{x^2}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\left(\frac{1}{x^2}\right)+\frac{\partial u}{\partial t}\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$=\ \frac{\partial u}{\partial r}\left(-\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\left(-\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{y^2}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(-\frac{1}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z^2}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial s}\left(-\frac{1}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z^2}\right)$$

Now

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0$$

12. **If** $u = f(x, y)$ *and* $x = rcos\theta,\ y = rsin\theta,$ **prove that** $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

Solution:

$$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta\,; \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

We have $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}.\cos\theta + \frac{\partial u}{\partial y}.\sin\theta$ ------(1)

Also we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}.(-r\sin\theta) + \frac{\partial u}{\partial y}.r\cos\theta$$

$$\therefore \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}.\sin\theta + \frac{\partial u}{\partial y}.\cos\theta \qquad \text{-------------(2)}$$

Squaring and adding (1) and (2), we get,

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(\frac{\partial u}{\partial y}\right)^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

13. **If** $z=f(x,y)$ *and* $x=u^2-v^2,\ y=2uv$, **prove that** $4(u^2+v^2)\left(z_{xx}+z_{yy}\right)=(z_{uu}+z_{vv})$

Solution:

$$x=u^2-v^2 \Rightarrow \frac{\partial x}{\partial u}=2u\ ,\frac{\partial x}{\partial v}=-2v \text{ and } y=2uv \Rightarrow \frac{\partial y}{\partial u}=2v\ ,\frac{\partial y}{\partial v}=2u$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(2u)+\frac{\partial z}{\partial y}(2v)$$

$$\frac{\partial^2 z}{\partial u^2}=(2u)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u}+\frac{\partial^2 z}{\partial y\partial x}(2u)\frac{\partial y}{\partial u}+2\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x\partial y}(2v)\frac{\partial x}{\partial u}+(2v)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v}$$

$$=4u^2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}(2uv)+2\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x\partial y}(2uv)+(4v^2)\frac{\partial^2 z}{\partial y^2}$$

$$=4u^2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}(4uv)+2\frac{\partial z}{\partial x}+(4v^2)\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-2v)+\frac{\partial z}{\partial y}(2u)=-2\frac{\partial z}{\partial x}(v)+\frac{\partial z}{\partial y}(2u)$$

$$\frac{\partial^2 z}{\partial v^2}=(-2v)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial v}+\frac{\partial^2 z}{\partial y\partial x}(-2v)\frac{\partial y}{\partial v}-2\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x\partial y}(2u)\frac{\partial x}{\partial v}+(2u)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v}$$

$$=4v^2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}(-2uv)-2\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x\partial y}(-2uv)+(4v^2)\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial v^2}\ =4v^2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y\partial x}(4uv)-2\frac{\partial z}{\partial x}+(4v^2)\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=4u^2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}(4uv)+2\frac{\partial z}{\partial x}+(4v^2)\frac{\partial^2 z}{\partial y^2}+4v^2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y\partial x}(4uv)-2\frac{\partial z}{\partial x}+(4v^2)\frac{\partial^2 z}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(4u^2 + 4v^2\right) + \frac{\partial^2 z}{\partial y^2}\left(4u^2 + 4v^2\right)$$

$$= 4\left(u^2 + v^2\right)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 4\left(u^2 + v^2\right)(z_{xx} + z_{yy})$$

$$(z_{uu} + z_{vv}) = 4\left(u^2 + v^2\right)(z_{xx} + z_{yy})$$

14. **If** $z = f(x, y)$ *and* $x = e^u \sin v,\ y = e^u \cos v,$ **prove that**

$$z_{xx} + z_{yy} = (x^2 + y^2)(z_{uu} + z_{vv})$$

Solution:

$$x = e^u \sin v \Rightarrow \frac{\partial x}{\partial u} = e^u \sin v \ , \frac{\partial x}{\partial v} = e^u \cos v$$

$$y = e^u \cos v \Rightarrow \frac{\partial y}{\partial u} = e^u \cos v \ , \frac{\partial y}{\partial v} = -e^u \sin v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} e^u \sin v + \frac{\partial z}{\partial y} e^u \cos v$$

$$\frac{\partial^2 z}{\partial u^2} = (e^u \sin v)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x}(e^u \sin v)\frac{\partial y}{\partial u} + \frac{\partial z}{\partial x} e^u \sin v$$

$$+ \frac{\partial^2 z}{\partial x \partial y}(e^u \cos v)\frac{\partial x}{\partial u} + (e^u \cos v)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y} e^u \cos v$$

$$\frac{\partial^2 z}{\partial u^2} = e^{2u}\sin^2 u\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y\partial x}(e^{2u}\sin v\cos v) + e^u\left(\sin v\frac{\partial z}{\partial x} + \cos v\frac{\partial z}{\partial y}\right) + (e^{2u}\cos^2 v)\frac{\partial^2 z}{\partial y^2} \quad ..(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}e^u\cos v - \frac{\partial z}{\partial y}e^u\sin v$$

$$\frac{\partial^2 z}{\partial v^2} = (e^u\cos v)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 z}{\partial y\partial x}(e^u\cos v)\frac{\partial y}{\partial v} + \frac{\partial z}{\partial x}(-e^u\sin v)$$

$$+\frac{\partial^2 z}{\partial x\partial y}(-e^u\sin v)\frac{\partial x}{\partial v} + (-e^u\sin v)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y}(-e^u\cos v)$$

$$\frac{\partial^2 z}{\partial v^2} = e^{2u}\cos^2 u\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial y\partial x}(e^{2u}\sin v\cos v) + e^u\left(-\sin v\frac{\partial z}{\partial x} - \cos v\frac{\partial z}{\partial y}\right) + (e^{2u}\sin^2 v)\frac{\partial^2 z}{\partial y^2} \quad ..(2)$$

Adding (1) and (2)

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\left(\frac{\partial^2 z}{\partial x^2}e^{2u}+\frac{\partial^2 z}{\partial y^2}e^{2u}\right)=e^{2u}(z_{xx}+z_{yy})$$

$$(z_{uu}+z_{vv})=e^{2u}(z_{xx}+z_{yy})$$

$$(z_{uu}+z_{vv})=(x^2+y^2)(z_{xx}+z_{yy})$$

JACOBIAN

If $u = u(x, y)$ and $v = v(x, y)$ are two functions of two independent variable x and y. Then the Jacobian of u & v is denoted by $J\left(\frac{u,v}{x,y}\right)$ *or* $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined by $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Note: if u, v and w are functions of three independent variables of x, y and z. Then their Jacobian is

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians

Property 1. If u and v are functions of x and y, then $\frac{\partial(u,v)}{\partial(x,y)} \mathrm{X} \frac{\partial(x,y)}{\partial(u,v)} = 1$

Property 2. (Chain Rule or Jacobian of Composite Function)

If u and v are functions of r and s, where r and s are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} X \frac{\partial(r,s)}{\partial(u,v)}$$

Property 3. If u, v, w are functionally dependent of a function x, y and z, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

Problems:

1. If $x = r\cos\theta$, $y = r\sin\theta$ then find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$

Solution:

Given $x = r\cos\theta$, $y = r\sin\theta$

Then $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$, $\dfrac{\partial y}{\partial r} = \sin\theta$, $\dfrac{\partial y}{\partial \theta} = r\cos\theta$

Now
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos^2\theta\right) = r(1) = r$$

2. **If** $x = uv,\ y = \frac{u}{v}$**, find** $\frac{\partial(x,y)}{\partial(u,v)}$

Solution: Given $x = uv\ , y = \frac{u}{v}$

Then $$\frac{\partial x}{\partial u} = v\ , \frac{\partial x}{\partial v} = u\ , \frac{\partial y}{\partial u} = \frac{1}{v}, \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = -\ \frac{2u}{v}$$

3. ***If*** $\boldsymbol{x = r\cos\theta}$ ***and*** $\boldsymbol{y = r\sin\theta}$***, then* find** $\boldsymbol{\frac{\partial r}{\partial x}}$**.**

Solution:

Given $x = r\cos\theta\ , y = r\sin\theta$

then $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$

Now $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

4. **If** $x = r\cos\theta\ , y = r\sin\theta , z = z$ **then find** $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Solution:

Given $x = r\cos\theta\ , y = r\sin\theta$

Then $\dfrac{\partial x}{\partial r} = \cos\theta\ , \dfrac{\partial x}{\partial \theta} = -r\sin\theta\ , \dfrac{\partial x}{\partial z} = 0 , \dfrac{\partial y}{\partial r} = \sin\theta , \dfrac{\partial y}{\partial \theta} = r\cos\theta , \dfrac{\partial y}{\partial z} = 0 , \dfrac{\partial z}{\partial r} = 0 , \dfrac{\partial z}{\partial \theta} = 0 , \dfrac{\partial z}{\partial z} = 1$

Now $$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta) = r$$

5. **If** $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ **then find** $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Solution : Given $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi\ ,\frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi\ ,\frac{\partial x}{\partial z} = -r\sin\theta\sin\phi,$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi\ ,\frac{\partial y}{\partial z} = r\sin\theta\cos\phi\ ,$$

$$\frac{\partial z}{\partial r} = \cos\theta,\frac{\partial z}{\partial\theta} = -r\sin\theta,\frac{\partial z}{\partial z} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(0 - (r\sin\theta\cos\phi)\cos\theta)$$

$$-r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\phi\cos^2\theta + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta(\cos^2\phi + \sin^2\phi) + r^2\sin\theta\cos^2\theta(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta(\sin^2\theta + \cos^2\theta) = r^2\sin\theta$$

6. **If** $u = x + y + z,\ uv = y + z,\ uvw = z,$ **show that** $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \mathrm{u}^2 v$

Solution: Given

$$u = x + y + z \quad -(1) \quad uv = y + z \quad -(2) \quad uvw = z \quad -(3)$$

Using (2) in (1), we get, $x = u - (y + z) = u - uv = u(1 - v)$

$$x = u(1 - v)\text{.......}(4)$$

Using (3) in (2) we get, $\mathrm{y} = uv - \mathrm{z} = \mathrm{uv} - uvw = uv(1 - w)$

$$\mathrm{y} = uv(1 - w)\text{.......}(5)$$

From (4) $\dfrac{\partial \mathrm{x}}{\partial \mathrm{u}} = 1 - \mathrm{v}, \quad \dfrac{\partial \mathrm{x}}{\partial \mathrm{v}} = -\mathrm{u}, \quad \dfrac{\partial \mathrm{x}}{\partial \mathrm{w}} = 0$

From (5) $\dfrac{\partial \mathrm{y}}{\partial \mathrm{u}} = \mathrm{v}.(1 - \mathrm{w}), \quad \dfrac{\partial \mathrm{y}}{\partial \mathrm{v}} = \mathrm{u}.(1 - \mathrm{w}), \quad \dfrac{\partial \mathrm{y}}{\partial \mathrm{w}} = -\mathrm{uv}$

From (3) $\dfrac{\partial \mathrm{z}}{\partial \mathrm{u}} = \mathrm{vw}, \quad \dfrac{\partial \mathrm{z}}{\partial \mathrm{v}} = \mathrm{uw}, \quad \dfrac{\partial \mathrm{z}}{\partial \mathrm{w}} = \mathrm{uv}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1\text{-v} & \text{-u} & 0 \\ \text{v(1-w)} & \text{u(1-w)} & -uv \\ \text{vw} & \text{wu} & \text{uv} \end{vmatrix}$$

$$= (1-v)\left[u^2v\,(1-w) + u^2vw\right] + u\left[uv^2(1-w) + uv^2w\right]$$

$$= (1-v)u^2v + u^2v^2 = u^2v$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \text{u}^2\text{v}$$

7. **Find the Jacobian of $\mathbf{y_1, y_2, y_3}$ with respect to** x_1, x_2, x_3 **if** $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_1 x_3}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$;

Given

$$y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_1 x_3}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{{x_1}^2}, \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \frac{\partial y_2}{\partial x_2} = -\frac{x_1 x_3}{{x_2}^2}, \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_2}{x_3}, \frac{\partial y_2}{\partial x_2} = \frac{x_1}{x_3}, \frac{\partial y_2}{\partial x_3} = -\frac{x_1 x_2}{{x_3}^2}$$

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1{}^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2{}^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3{}^2} \end{vmatrix}$$

Taking $\frac{1}{x_1}$ from Row 1 , $\frac{1}{x_2}$ from Row 2 and $\frac{1}{x_3}$ from Row 3,we get

$$= \frac{1}{x_1x_2x_3} \begin{vmatrix} -\frac{x_2x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_1x_3}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1x_2}{x_3} \end{vmatrix}$$

$$= \frac{1}{x_1{}^2 x_2{}^2 x_3{}^2} \begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1{}^2 x_2{}^2 x_3{}^2}{x_1{}^2 x_2{}^2 x_3{}^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1-1) - 1(-1-1) + 1(1+1) = 4$$

$$\therefore \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$$

8. **If** $x = r\cos\theta\ , y = r\sin\theta$ **verify that** $\frac{\partial(x,y)}{\partial(r,\theta)} \mathrm{X} \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Given $x = r\cos\theta\ , y = r\sin\theta$

Then $\frac{\partial x}{\partial r} = \cos\theta\ , \frac{\partial x}{\partial \theta} = -r\sin\theta\ , \frac{\partial y}{\partial r} = \sin\theta , \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\text{Now} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos{}^2\theta\right) = r(1) = r$$

Now expressing and θ in terms of x and y

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}; \frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}(2y) = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}; \quad \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{1}{r^3}(x^2+y^2) = \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \mathrm{X} \frac{\partial(r,\theta)}{\partial(x,y)} = r \, \mathrm{X} \, \frac{1}{r} = 1$$

9. **If** $u = 2xy, v = x^2 - y^2, x = r\cos\theta \ , y = r\sin\theta,$ compute $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

Solution :

Given $u = 2xy, v = x^2 - y^2,$

$$\frac{\partial(u,u)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2 = -4(x^2 + y^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos^2\theta\right) = r(1) = r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \text{X} \frac{\partial(x,y)}{\partial(r,\theta)} = -4r(x^2 + y^2) = -4r^3 \ \ (\sin ce\ x^2 + y^2 = r^2)$$

10. **Prove that the function** $u=\frac{x+y}{x-y}, v=\frac{xy}{(x-y)^2}$ **are functionally dependent**

Solution: If u and v are functionally dependent, then their $\frac{\partial(u,u)}{\partial(x,y)}=0$

Given $u=\frac{x+y}{x-y}, v=\frac{xy}{(x-y)^2}$

Then $\frac{\partial u}{\partial x}=\frac{(x-y)-(x+y)}{(x-y)^2}=\frac{-2y}{(x-y)^2}$

$$\frac{\partial u}{\partial y}=\frac{(x-y)-(x+y)(-1)}{(x-y)^2}=\frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x}=\frac{(x-y)^2 y-2xy(x-y)}{(x-y)^4}=\frac{y(x-y)\left[x-y-2x\right]}{(x-y)^3}$$

$$\frac{\partial v}{\partial x}=\frac{-y(x+y)}{(x-y)^3}$$

$$\frac{\partial v}{\partial y}=\frac{(x-y)^2-2xy(x-y)(-1)}{(x-y)^4}=\frac{(x-y)\left[x-y+2xy\right]}{(x-y)^4}$$

$$=\frac{x(x+y)}{(x-y)^3}$$

$$\frac{\partial(u,u)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2}\\ \frac{-y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3}\end{vmatrix}=-\frac{2xy(x+y)}{(x-y)^5}+\frac{2xy(x+y)}{(x-y)^5}=0$$

Therefore u and v are functionally dependent.

11. **If** $u = xy + yz + zx, v = x^2 + y^2 + z^2, w = x + y + z$

determine the functional relationship between u, v, w.

Solution:

$$u = xy + yz + zx \quad \Rightarrow \frac{\partial \mathrm{u}}{\partial x} = y + z,\ \frac{\partial u}{\partial y} = x + z,\ \frac{\partial u}{\partial z} = x + y$$

$$v = x^2 + y^2 + z^2 \quad \Rightarrow \frac{\partial \mathrm{v}}{\partial x} = 2x,\ \frac{\partial \mathrm{v}}{\partial y} = 2y,\ \frac{\partial \mathrm{v}}{\partial z} = 2z$$

$$w = x + y + z \quad \Rightarrow \frac{\partial \mathrm{w}}{\partial x} = 1,\ \frac{\partial \mathrm{w}}{\partial y} = 1,\ \frac{\partial \mathrm{w}}{\partial z} = 1$$

Hence,

$$\frac{\partial(u,\mathrm{v},\mathrm{w})}{\partial(x,y,z)}=\begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$=2(y+z)(y-z)-2(x-z)2(x+z)+2(y+x)(y-x)=0$$

Therefore u , v and w are functionally dependent.

The relation is

$$w^2=(\mathrm{x}+\mathrm{y}+\mathrm{z})^2==\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2+2(\mathrm{xy}+\mathrm{yz}+\mathrm{zx})=\mathrm{v}+2\mathrm{u}.$$

12. **If** $u=\sin^{-1}x+\sin^{-1}y, v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

determine the functional relationship between u, v.

Solution :

Given $u=\sin^{-1}x+\sin^{-1}y, v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-x^2}}\ ;\ \frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-y^2}}\ ;$$

$$\frac{\partial v}{\partial x}=\sqrt{1-y^2}+\frac{-xy}{\sqrt{1-x^2}};\frac{\partial v}{\partial x}=\sqrt{1-x^2}+\frac{-xy}{\sqrt{1-x^2}}$$

$$\frac{\partial(u,u)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}}\\ \sqrt{1-y^2}+\frac{-xy}{\sqrt{1-x^2}} & \sqrt{1-x^2}+\frac{-xy}{\sqrt{1-y^2}}\end{vmatrix}$$

$$= \left(1 + \frac{-xy}{\left(\sqrt{1-y^2}\right)\left(\sqrt{1-x^2}\right)}\right) - \left(1 - \frac{xy}{\left(\sqrt{1-y^2}\right)\left(\sqrt{1-x^2}\right)}\right) = 0$$

Therefore u, v are functionally dependent.

Take $x = \sin\alpha, y = \sin\beta \Rightarrow \alpha = \sin^{-1}(x), \beta = \sin^{-1}(y)$

Now $u = \sin^{-1} x + \sin^{-1} y = \alpha + \beta$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$$

$$= \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta)$$

$$= \sin u$$

TAYLOR'S SERIES

TAYLOR'S SERIES FORMULA

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}$$

$+\ldots\ldots$ is called the Taylor's series at the point (a,b)

When a=0 and b=0, the above series is called **MacLaurin's series**

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2 f_{xx}(0,0)+2xy\,f_{xy}(0,0)+y^2 f_{yy}(0,0)\right]+$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0)+3x^2y\,f_{xxy}(0,0)+3xy^2 f_{xyy}(0,0)+y^3 f_{yyy}(0,0)\right]$$

$+\ldots\ldots$

Problems:

1. **Expand** $e^x \sin y$ **as Maclaurin's series**

Solution:

Given $f(x,y) = e^x \sin y$ and here a= b=0

We use Maclaurin's formula

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$
$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]+\ldots$$

Function	At the Point (0,0)
$f(x, y) = e^x \sin y$	$f(0,0) = e^0 \sin 0 = 0$
$f_x(x, y) = e^x \sin y$	$f_x(0,0) = e^0 \sin 0 = 0$
$f_{xx}(x, y) = e^x \sin y$	$f_{xx}(0,0) = e^0 \sin 0 = 0$
$f_{xxx}(x, y) = e^x \sin y$	$f_{xxx}(0,0) = e^0 \sin 0 = 0$
$f_y(x, y) = e^x \cos y$	$f_y(0,0) = e^0 \cos 0 = 1$
$f_{yy}(x, y) = -e^x \sin y$	$f_{yy}(0,0) = -e^0 \sin 0 = 0$
$f_{yyy}(x, y) = -e^x \cos y$	$f_{yyy}(0,0) = -e^0 \cos 0 = -1$
$f_{xy}(x, y) = e^x \cos y$	$f_{xy}(0,0) = e^0 \cos 0 = 1$

$$f_{xxy}(x,y)=e^{x}\cos y \qquad f_{xxy}(0,0)=e^{0}\cos 0=1$$

$$f_{xyy}(x,y)=-e^{x}\sin y \qquad f_{xyy}(0,0)=-e^{0}\sin 0=0$$

$$e^{x}\sin y=0+x(0)+y(1)+\frac{1}{2!}\left[x^{2}(0)+2xy(1)+y^{2}(0)\right]+\frac{1}{3!}\left[x^{3}(0)+3x^{2}y(1)+3xy^{2}(0)+y^{3}(-1)\right]+\ldots$$

$$=y+xy+\frac{x^{2}y}{2}-\frac{y^{3}}{6}+\ldots$$

2. **Expand e^{xy} in powers of x and y up to third degree**

Solution:

Given $f(x,y)=e^{xy}$ and here a= b=0. We use Maclaurin's formula

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xy\,f_{xy}(0,0)+y^2f_{yy}(0,0)\right]+$$

$$+\frac{1}{3!}\left[x^3f_{xxx}(0,0)+3x^2y\,f_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\right]+\ldots$$

Function	At the Point (0,0)
$f(x,y)=e^{xy}$	$f(0,0)=e^{0}=1$
$f_x(x,y)=ye^{xy}$	$f_x(0,0)=0$
$f_{xx}(x,y)=y^2e^{xy}$	$f_{xx}(0,0)=0$
$f_{xxx}(x,y)=y^3e^{xy}$	$f_{xxx}(0,0)=0$
$f_y(x,y)=xe^{xy}$	$f_y(0,0)=0$
$f_{yy}(x,y)=x^2e^{xy}$	$f_{yy}(0,0)=0$
$f_{yyy}(x,y)=x^3e^{xy}$	$f_{yyy}(0,0)=0$
$f_{xy}(x,y)=e^{xy}+x^2e^{xy}$	$f_{xy}(0,0)=1+0=1$
$f_{xxy}(x,y)=e^{xy}y+2xe^{xy}+x^2ye^{xy}$	$f_{xxy}(0,0)=0$
$f_{xyy}(x,y)=e^{xy}2x+x^2e^{xy}y$	$f_{xyy}(0,0)=0$

$$e^{xy} = 1 + x(0) + y(0) + \frac{1}{2!}\left[x^2(0) + 2xy\,(1) + y^2(0)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2y\,(0) + 3xy^2(0) + y^3(0)\right] + \ldots$$

$$= 1 + xy + \ldots.$$

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3. **Expand** $e^x \log(1+y)$ **in powers of x and y up to third degree**

Solution:

Given $f(x,y) = e^x \log(1+y)$ and here

We use Maclaurin's formula

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots$$

Function	At the Point (0,0)
$f(x,y)=e^x\log(1+y)$	$f(0,0)=e^0(\log 1)=0$
$f_x(x,y)=e^x\log(1+y)$	$f_x(0,0)=e^0(\log 1)=0$
$f_{xx}(x,y)=e^x\log(1+y)$	$f_{xx}(0,0)=e^0(\log 1)=0$
$f_{xxx}(x,y)=e^x\log(1+y)$	$f_{xxx}(0,0)=e^0(\log 1)=0$
$f_y(x,y)=\dfrac{e^x}{1+y}$	$f_y(0,0)=\dfrac{e^0}{1+0}=1$
$f_{yy}(x,y)=-\dfrac{e^x}{(1+y)^2}$	$f_{yy}(0,0)=-\dfrac{e^0}{(1+0)^2}=-1$
$f_{yyy}(x,y)=-\dfrac{2e^x}{(1+y)^3}$	$f_{yyy}(0,0)=-\dfrac{2e^0}{(1+0)^3}=-2$

$$f_{xy}(x,y)=\frac{e^{x}}{(1+y)}$$

$$f_{xxy}(x,y)=\frac{e^{x}}{(1+y)}$$

$$f_{xyy}(x,y)=-\frac{e^{x}}{(1+y)^{2}}$$

$$f_{xy}(0,0)=\frac{e^{0}}{1+0}=1$$

$$f_{xxy}(0,0)=\frac{e^{0}}{1+0}=1$$

$$f_{xyy}(0,0)=-\frac{e^{0}}{(1+0)^{2}}=-1$$

$$f(x,y)=f(0,0)+xf_{x}(0,0)+yf_{y}(0,0)+\frac{1}{2!}\left[x^{2}f_{xx}(0,0)+2xy\,f_{xy}(0,0)+y^{2}f_{yy}(0,0)\right]+$$

$$+\frac{1}{3!}\left[x^{3}f_{xxx}(0,0)+3x^{2}y\,f_{xxy}(0,0)+3xy^{2}f_{xyy}(0,0)+y^{3}f_{yyy}(0,0)\right]$$

$$=y+\frac{1}{2!}\left(xy-y^{2}\right)+\frac{1}{3!}\left(x^{2}y-xy^{2}+y^{3}\right)+\ldots\ldots.$$

4. **Expand** $e^x \cos y$ **in powers of** $(x-1) \text{ and } \left(y-\frac{\pi}{4}\right)$ **up to third degree**

Solution:

Given $f(x,y)=e^x \cos y$ and here $a=1, b=\frac{\pi}{4}$

$$f(x,y) = f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)\\ +3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\end{array}\right]+\ldots$$

Function	At the Point $\left(1,\frac{\pi}{4}\right)$
$f(x,y)=e^x\cos y$	$f\left(1,\frac{\pi}{4}\right)=e^1\cos\left(\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}$
$f_x(x,y)=e^x\cos y$	$f_x\left(1,\frac{\pi}{4}\right)=e^1\cos\left(\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}$
$f_{xx}(x,y)=e^x\cos y$	$f_{xx}\left(1,\frac{\pi}{4}\right)=e^1\cos\left(\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}$
$f_{xxx}(x,y)=e^x\cos y$	$f_{xxx}\left(1,\frac{\pi}{4}\right)=e^1\cos\left(\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}$
$f_y(x,y)=-e^x\sin y$	$f_y\left(1,\frac{\pi}{4}\right)=-e^1\sin\left(\frac{\pi}{4}\right)=-\frac{e}{\sqrt{2}}$
$f_{yy}(x,y)=-e^x\cos y$	$f_{yy}\left(1,\frac{\pi}{4}\right)=-e^1\cos\left(\frac{\pi}{4}\right)=-\frac{e}{\sqrt{2}}$
$f_{yyy}(x,y)=e^x\sin y$	$f_{yyy}\left(1,\frac{\pi}{4}\right)=e^1\sin\left(\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}$

$$f_{xy}(x,y) = -xe^{x}\sin y$$

$$f_{xxy}(x,y) = -x^{2}e^{x}\sin y$$

$$f_{xyy}(x,y) = -xe^{x}\cos y$$

$$f_{xy}\left(1,\frac{\pi}{4}\right) = -1.e^{1}\sin\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{xxy}\left(1,\frac{\pi}{4}\right) = -1.e^{1}\sin\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{xyy}\left(1,\frac{\pi}{4}\right) = -1.e^{1}\cos\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$\begin{aligned} f(x,y) = {} & f\left(1,\frac{\pi}{4}\right) + (x-1)\,f_{x}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)f_{y}\left(1,\frac{\pi}{4}\right) \\ & + \frac{1}{2!}\left[(x-1)^{2}\,f_{xx}\left(1,\frac{\pi}{4}\right) + 2(x-1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^{2} f_{yy}\left(1,\frac{\pi}{4}\right)\right] \\ & + \frac{1}{3!}\left[\begin{array}{l} (x-1)^{3}\,f_{xxx}\left(1,\frac{\pi}{4}\right) + 3(x-1)^{2}\left(y-\frac{\pi}{4}\right)f_{xxy}\left(1,\frac{\pi}{4}\right) \\ +3(x-1)\left(y-\frac{\pi}{4}\right)^{2} f_{xyy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^{3} f_{yyy}\left(1,\frac{\pi}{4}\right) \end{array}\right] + \ldots \end{aligned}$$

$$f(x,y)=\frac{e}{\sqrt{2}}\left(\begin{aligned}&1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{1}{2!}\left[(x-1)^2-2(x-1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2\right]\\&+\frac{1}{3!}\left[(x-1)^3-3(x-1)^2\left(y-\frac{\pi}{4}\right)-3(x-1)\left(y-\frac{\pi}{4}\right)^2+\left(y-\frac{\pi}{4}\right)^3\right]\end{aligned}\right)+\ldots$$

---

5. **Expand** $x^2y+3y-2$ **in powers of** $(x-1)$ **and** $(y+2)$ **using Taylor's theorem**

Solution:

Given $f(x,y)=x^2y+3y-2$ and here $a=1,b=-2$

$$\begin{aligned}f(x,y)=\ &f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)\\&+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]\\&+\frac{1}{3!}\left[\begin{aligned}&(x-a)^3f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)\\&+3(x-a)(y-b)^2f_{xyy}(a,b)+(y-b)^3f_{yyy}(a,b)\end{aligned}\right]+\ldots\end{aligned}$$

Function	At the Point(1,-2)
$f(x,y)=x^2y+3y-2$	$f(1,-2)=-2-6-2=-10$
$f_x(x,y)=2xy$	$f_x(1,-2)=-4$
$f_{xx}(x,y)=2y$	$f_{xx}(1,-2)=-4$
$f_{xxx}(x,y)=0$	$f_{xxx}(1,-2)=0$
$f_y(x,y)=x^2+3$	$f_y(1,-2)=4$
$f_{yy}(x,y)=0$	$f_{yy}(1,-2)=0$
$f_{yyy}(x,y)=0$	$f_{yyy}(1,-2)=0$
$f_{xy}(x,y)=2x$	$f_{xy}(1,-2)=2$
$f_{xxy}(x,y)=2$	$f_{xxy}(1,-2)=2$
$f_{xyy}(x,y)=0$	$f_{xyy}(0,0)=0$

$$
\begin{aligned}
f(x,y) = {} & f(1,-2) + (x-1) f_x(1,-2) + (y+2) f_y(1,-2) \\
& + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2) \right] \\
& + \frac{1}{3!}\left[\begin{array}{l} (x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2 (y+2) f_{xxy}(1,-2) \\ +3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2) \end{array} \right] + \ldots
\end{aligned}
$$

$$
\begin{aligned}
x^2y+3y-2 &= -10-4(x-1)+4(y+2)+\frac{1}{2!}\left[(-4)(x-1)^2+4(x-1)(y+2)\right]+\frac{1}{3!}\left[6(x-1)^2(y+2)\right]+\ldots \\
&= -10-4(x-1)+4(y+2)+\left[(-2)(x-1)^2+2(x-1)(y+2)\right]+\left[(x-1)^2(y+2)\right]+\ldots
\end{aligned}
$$

6. **Expand $x^2y^2+2x^2y+3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's theorem**

Solution:

Given $f(x,y)=x^2y^2+2x^2y+3xy^2$ and here $a=-2, b=1$

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)\\ +3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\end{array}\right]+\dots$$

Function	At the Point (2,-1)
$f(x,y)=x^2y^2+2x^2y+3\mathrm{xy}^2$	$f(-2,1)=4+8-6=6$
$f_x(x,y)=2xy^2+4xy+3y^2$	$f_x(-2,1)=-4-8+3=-9$
$f_{xx}(x,y)=2y^2+4y$	$f_{xx}(-2,1)=6$
$f_{xxx}(x,y)=0$	$f_{xxx}(-2,1)=0$
$f_y(x,y)=2x^2y+2x^2+6xy$	$f_y(-2,1)=4$
$f_{yy}(x,y)=2x^2+6x$	$f_{yy}(-2,1)=-4$
$f_{yyy}(x,y)=0$	$f_{yyy}(-2,1)=0$
$f_{xy}(x,y)=4xy+6y+4x$	$f_{xy}(-2,1)=-10$
$f_{xxy}(x,y)=4y+4$	$f_{xxy}(-2,1)=8$
$f_{xyy}(x,y)=4x+6$	$f_{xyy}(-2,1)=-2$

$$
\begin{aligned}
f(x,y) = {} & f(2,-1) + (x+2) f_x(-2,1) + (y-1) f_y(-2,1) \\
& + \frac{1}{2!}\left[(x+2)^2 f_{xx}(-2,1) + 2(x+2)(y-1) f_{xy}(-2,1) + (y-1)^2 f_{yy}(-2,1) \right] \\
& + \frac{1}{3!}\left[\begin{array}{l} (x+2)^3 f_{xxx}(-2,1) + 3(x+2)^2(y-1) f_{xxy}(-2,1) \\ +3(x+2)(y-1)^2 f_{xyy}(-2,1) + (y-1)^3 f_{yyy}(-2,1) \end{array} \right] + \dots
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) = {} & 6 + (x+2)(-9) + (y-1)(4) + \frac{1}{2!}\left[(x+2)^2(6) + 2(x+2)(y-1)(-10) + (y-1)^2(-4) \right] \\
& + \frac{1}{3!}\left[(x+2)^3(0) + 3(x+2)^2(y-1)(8) + 3(x+2)(y-1)^2(-2) + (y-1)^3(0) \right] + \dots \\
= {} & 6 - 9(x+2) + 4(y-1) + \left[3(x+2)^2 - 10(x+2)(y-1) - 2(y-1)^2 \right] \\
& + \left[(x+2)^2(y-1)(4) - 3(x+2)(y-1)^2 \right] + \dots
\end{aligned}
$$

7. **Expand** $\tan^{-1}\left(\frac{y}{x}\right)$ **at the point (1,1) up to second degree**

Solution:

Given $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ and here $a=1, b=1$

$$f(x,y) = f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\dots$$

Function	At the Point (1,1)
$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$	$f(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$
$f_x(x,y) = \frac{-y}{x^2+y^2}$	$f_x(1,1) = -\frac{1}{2}$
$f_{xx}(x,y) = \frac{2xy}{(x^2+y^2)^2}$	$f_{xx}(1,1) = \frac{1}{2}$
$f_y(x,y) = \frac{x}{x^2+y^2}$	$f_y(1,1) = \frac{1}{2}$
$f_{yy}(x,y) = -\frac{2xy}{(x^2+y^2)^2}$	$f_{yy}(1,1) = -\frac{1}{2}$
$f_{xy}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$	$f_{xy}(1,1) = 0$

$$f(x,y)=f(1,1)+\Big[(x-1)f_x(1,1)+(y-1)f_y(1,1)\Big]$$

$$+\frac{1}{2!}\Big[(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)\Big]+\dots$$

$$=\frac{\pi}{4}+\frac{1}{2}\big((y-1)-(x-1)\big)+\frac{1}{2}\big((x-1)^2-(y-1)^2\big)+\dots$$

$$=\frac{\pi}{4}+\frac{1}{2}(y-x)+\frac{1}{2}\big(x^2-y^2+2(x-y)\big)+\dots$$

8. **Expand the function $\sin(xy)$ at $\left(1, \frac{\pi}{2}\right)$ as a Taylor series up to second degree**

Solution:

Given $f(x, y) = \sin(xy)$ and here $a = 1, b = \frac{\pi}{2}$

$$f(x, y) = f(a, b) + \left[(x - a) f_x(a, b) + (y - b) f_y(a, b)\right]$$

$$+ \frac{1}{2!}\left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\right] + \ldots$$

Function	At the Point $\left(1, \frac{\pi}{2}\right)$
$f(x,y)=\sin xy$	$f\left(1,\frac{\pi}{2}\right)=1$
$f_x(x,y)=y\cos(xy)$	$f_x\left(1,\frac{\pi}{2}\right)=0$
$f_y(x,y)=x\cos(xy)$	$f_y\left(1,\frac{\pi}{2}\right)=0$
$f_{xx}(x,y)=-y^2\sin(xy)$	$f_{xx}\left(1,\frac{\pi}{2}\right)=-\frac{\pi^2}{4}$
$f_{xy}(x,y)=-xy\sin(xy)+\cos(xy)$	$f_{xy}\left(1,\frac{\pi}{2}\right)=-\frac{\pi}{2}$
$f_{yy}(x,y)=-x^2\sin(xy)$	$f_{yy}\left(1,\frac{\pi}{2}\right)=-1$

$$f(x,y) = f\left(1,\frac{\pi}{2}\right) + \left[(x-1)f_x\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(1,\frac{\pi}{2}\right)\right] + \dots$$

$$= 1 + \frac{1}{2}\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right] + \cdots$$

Maxima and Minima :

Definition:

A function $f(x,y)$ is said to have a relative maximum (or maximum) at (a,b) if $f(a,b) > f(a+h,b+k)$ for small values of h and k.

A function $f(x,y)$ is said to have a relative minimum (or minimum) at (a,b) if $f(a,b) < f(a+h,b+k)$ for small values of h and k.

Note:

A maximum or minimum value of a function is called as its extreme value.

Maxima and Minima of a function of two variables

Notation: $p = \frac{\partial f}{\partial x}; q = \frac{\partial f}{\partial y}; r = \frac{\partial^2 f}{\partial x^2}; s = \frac{\partial^2 f}{\partial x \partial y}; t = \frac{\partial^2 f}{\partial y^2}$

Working rule:

Let f(x, y) be the given function

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
2. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Solution of the equations are stationary point
3. Find the value of r, s, t and $rt-s^2$ at all the stationary points.

r or t	**$rt-s^2$**	**Conclusion**
$r < 0$	$(rt-s^2) > 0$	f(x, y) attains its maximum at that stationary point
$r > 0$	$(rt-s^2) > 0$	f(x, y) attains its minimum at that stationary point
-	$(rt-s^2) < 0$	Neither maximum nor minimum. The stationary point is saddle point
-	$(rt-s^2) = 0$	Further investigation needed

PROBLEMS:

1. **Find the maximum and minimum value for the function** $f(x,y)=x^2+y^2+6x+12$

Solution:

Let $f(x,y) = x^2 + y^2 + 6x + 12$

$$p = \frac{\partial f}{\partial x} = 2x + 6; q = \frac{\partial f}{\partial y} = 2y; r = \frac{\partial^2 f}{\partial x^2} = 2;\ \ s = \frac{\partial^2 f}{\partial x \partial y} = 0\ andt = \frac{\partial^2 f}{\partial y^2} = 2.$$

p = 0 and q = 0 implies x = –3 and y = 0.

Therefore the stationary point is (–3, 0).

At (–3, 0), r = 2 > 0 and rt – s^2 = 4 > 0.

Therefore f(x, y) obtains its minimum value at (–3, 0).

The minimum value is f(–3, 0) = 3.

2. Find the maximum and minimum of the function $3(x^2 - y^2) - x^3 + y^3$

Solution:

Let $f(x, y) = 3(x^2 - y^2) - x^3 + y^3$

$$p = \frac{\partial f}{\partial x} = 6x - 3x^2; q = \frac{\partial f}{\partial y} = -6y + 3y^2$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6 - 6x; \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0 \ \ and$$

$$t = \frac{\partial^2 f}{\partial y^2} = -6 + 6y$$

p = 0 implies x = 0 and x = 2.

and q = 0 implies y = 0 and y = 2

Therefore the stationary points are (0, 0), (0, 2), (2, 0) and (2, 2).

At stationary points	**$r = 6 – 6x$**	**$rt – s^2$**	**Conclusion**	**Extreme value**
(0, 0)	6	–36	Saddle point	–
(0, 2)	6	36	Minimum	f(0, 2) = – 4
(2, 0)	–6	36	Maximum	f(2, 0) = 4
(2, 2)	–6	–36	Saddle point	–

Thus f(x, y) obtains its maximum at (2, 0) and the maximum value is 4.

Similarly, f(x, y) obtains its minimum at (0, 2) and the minimum value is -4.

3. **Find the maximum and minimum of the function $\boldsymbol{x^3 + y^3 - 12x - 3y + 20}$**

Solution:

Let $f(x, y) = x^3 + y^3 - 12x - 3y + 20$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 12; q = \frac{\partial f}{\partial y} = 3y^2 - 3;$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x; \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0 \; and \; \; t = \frac{\partial^2 f}{\partial y^2} = 6y.$$

p = 0 implies x = –2 and x = 2.

and q = 0 implies y = –1 and y = 1

Therefore the stationary points are (–2, –1), (–2, 1), (2, –1) and (2, 1).

At stationary points	**$r = 6x$**	**$rt - s^2$**	**Conclusion**	**Extreme value**
$(-2, -1)$	-12	72	Maximum	$f(-2,-1) = 38$
$(-2, 1)$	-12	-72	Saddle point	–
$(2, -1)$	12	-72	Saddle point	–
$(2, 1)$	12	72	Minimum	$f(2, 1) = 2$

Thus $f(x, y)$ obtains its maximum at $(-2, -1)$ and the maximum value is 38.

Similarly, $f(x, y)$ obtains its minimum at $(2, 1)$ and the minimum value is 2.

4. Find the maximum and minimum values of $f(x, y) = x^3 + y^3 - 3axy$.

Solution:

Let $f(x, y) = x^3 + y^3 - 3axy$

$p = f_x = 3x^2 - 3ay$; $q = f_y = 3y^2 - 3ax$;

$r = f_{xx} = 6x$; $s = f_{xy} = -3a$; $t = f_{yy} = 6y$.

$p = 0$ and $q=0$ implies $3x^2 - 3ay = 0$and $3y^2 - 3ax = 0$

$x^2 = ay$ and $y^2 = ax$

i.e., $x^4 = a^2y^2$

$x^4 = a^3x$

i.e., $x(x^3 - a^3) = 0$

$\therefore x = 0$ or $x = a$

When $x = 0$, we get, $y = 0$ and when $x = a$, we get, $y = a$

$\therefore$ The stationary points are $(0,0)$ and (a, a)

At stationary points	**r**	**$rt - s^2$**	**Conclusion**	**Extreme value**
(0, 0)	0	$-9a^2 < 0$	Neither maximum nor minimum, Saddle point	-
(a, a)	6a	$27a^2$	If $a > 0$, then $r > 0$ and hence f(a, a) is a minimum value.	
			If $a < 0$, then $r < 0$ and hence f(a, a) is a maximum value.	

Thus the maximum or minimum value at (a, a) is $f(a, a) = -a^3$

5. Find the maxima or minima of $f(x, y) = 2(x - y)^2 - x^4 - y^4$

Solution:

Let $f(x,y) = 2(x - y)^2 - x^4 - y^4$

$p = f_x = 4(x - y) - 4x^3$; $q = f_y = - 4(x - y) - 4y^3$

$r = f_{xx} = 4 - 12x^2$; $s = f_{xy} = - 4$; $t = f_{yy} = 4 - 12y^2$

solving $p = 0$ and $q = 0$ implies $x - y - x^3 = 0$ →(1)

and $-(x - y) - y^3 = 0$ →(2)

Adding (1) and (2) $x^3 + y^3 = 0$
i.e., $(x + y)(x^2 - xy + y^2) = 0$
∴ $x = -y$ or $x^2 - xy + y^2 = 0$ (Check: $x^2 - xy + y^2 > 0$, always)

Putting in (1) $x = -y$, we get,
$-2y + y^3 = 0$
i.e., $y(y^2 - 2) = 0$

i.e., $y = 0, \sqrt{2}, -\sqrt{2}$

The corresponding x values are $0, -\sqrt{2}, \sqrt{2}$

$\therefore$ The stationary points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

At stationary points	**$r=4-12x^2$**	**$rt - s^2$**	**Conclusion**	**Extreme value**
$(0, 0)$	4	0	Further investigation needed	-
$(\sqrt{2}, -\sqrt{2})$	–20	384	Maximum	$f(\sqrt{2}, -\sqrt{2}) = 8$
$(-\sqrt{2}, \sqrt{2})$	–20	384	Maximum	$f(-\sqrt{2}, \sqrt{2}) = 8$

6. **Find the maxima or minima of $f(x, y) = x^2y^2(1 - x - y)$**

Solution:

Let $f(x,y) = x^2y^2(1 - x - y)$

$$p=\frac{\partial f}{\partial x} = 2xy^2(1 - x - y) + x^2y^2(-1) = xy^2(2 - 3x - 2y\Big)$$

$$q=\frac{\partial f}{\partial y} = 2x^2y(1 - x - y) + x^2y^2(-1) = x^2y(2 - 2x - 3y\Big)$$

solving p = 0 and q = 0 implies

$$xy^2(2 - 3x - 2y) = 0 \text{ ------------- -}(1)$$

$$x^2y(2 - 2x - 3y) = 0 \text{ ------------- - }(2)$$

(1) and (2) $\Rightarrow$

$x=0$, $y=0$ and

$$(2-3x-2y)=0 ---------(3)$$

$$(2-2x-3y)=0 ---------(4)$$

solving (3) and (4)

$2x+3y=2$

$3x+2y=2$

$9y-4y=2 \Rightarrow 5y=2 \Rightarrow y=\frac{2}{5}$

$2x=2-3y == 2-\frac{6}{5} = \frac{4}{5} \Rightarrow x=\frac{2}{5}$

$\therefore$ $The\ Stationary\ pts\ are$ $(0,0), \left(\frac{2}{5},\frac{2}{5}\right), (0,1), \left(0,\frac{2}{3}\right), \left(\frac{2}{3},0\right), (1,0)$

$$A = f_{xx} = -6xy^2 + 2y^2 - 2y^3$$

$$B = f_{xy} = x^2y(-2) + 2y(2 - 2x - 3y)$$

$$C = f_{yy} = x^2y(-3) + (2 - 2x - 3y)x^2$$

At stationary points	**$r=4-12x^2$**	**$rt-s^2$**	**Conclusion**	**Extreme value**
(0, 0)	0	0	Further investigation needed	-
(0,1)	0	0	Further investigation needed	-
$(0,\frac{2}{3})$	$\frac{8}{27}$	0	Further investigation needed	-
$(\frac{2}{3},0)$	0	0	Further investigation needed	-
(1,0)	0	0	Further investigation needed	-
$(\frac{2}{5},\frac{2}{5})$	$\frac{-24}{125}$	+ ve	the maximum	$\frac{16}{3125}$

Constrained maximum and minimum– Lagrange’s multipliers methods

Let f(x, y, z) = 0 be the function whose extreme values should be found subject to the condition (constraint) ϕ (x, y, z) = 0.

We define F(x, y, z) = f(x, y, z) + $\lambda\phi$ (x, y, z), where λ is called Lagrange multiplier.

For extreme values, solve $\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0; \frac{\partial F}{\partial \lambda} = 0$

PROBLEMS :

1. **Find the maximum value of $x^m y^n z^p$ such that x + y + z = a**

Solution:

Given $f(x, y, z) = x^m y^n z^p$ and $\varphi(x, y, z) = x + y + z = a$

$$F(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a)$$

$$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda$$

$$\frac{\partial F}{\partial y} = n x^m y^{n-1} z^p + \lambda$$

$$\frac{\partial F}{\partial z} = p x^m y^n z^{p-1} + \lambda$$

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \lambda = mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\Rightarrow \frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} \quad = \quad \frac{m+n+p}{a}$$

$$\Rightarrow x = \frac{am}{m+n+p}; y = \frac{an}{m+n+p}; z = \frac{ap}{m+n+p}$$

Thus the maximum value of

$$F(x,y,z) = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p = \frac{a^{m+n+p}(m^m n^n p^p)}{(m+n+p)^{m+n+p}}$$

2. Find the minimum value of $x^2 + y^2 + z^2$ where $ax + by + cz = p$.

Solution:

We use Lagrange's method. Let $f(x, y, z) = x^2 + y^2 + z^2$.

$\varphi(x, y, z) = ax + by + cz - p$ and $F(x, y, z) = f(x, y, z) + \lambda\varphi(x, y, z)$
where λ is the Lagrange multiplier

Then $F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$

The stationary points are obtained by solving

$$F_x = 2x + a\lambda = 0 \tag{1}$$
$$F_y = 2y + b\lambda = 0 \tag{2}$$
$$F_z = 2z + c\lambda = 0 \tag{3}$$
and $$F_\lambda = ax + by + cz - p \tag{4}$$

From (1), $x = -\frac{a\lambda}{2}$

From (2), $y = -\frac{b\lambda}{2}$

From (3), $z = -\frac{c\lambda}{2}$

From (4), $a.\left(-\frac{a\lambda}{2}\right) + b.\left(-\frac{b\lambda}{2}\right) + c.\left(-\frac{c\lambda}{2}\right) = p$

$$\lambda = \frac{-2p}{a^2 + b^2 + c^2}$$

$$\therefore x = \frac{ap}{a^2+b^2+c^2}, \quad y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}$$

The only stationary point is $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$

The minimum value of

$$f(x,y,z)=\left(\frac{ap}{a^2+b^2+c^2}\right)^2+\left(\frac{bp}{a^2+b^2+c^2}\right)^2+\left(\frac{cp}{a^2+b^2+c^2}\right)^2$$

$$=\frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}=\frac{p^2}{a^2+b^2+c^2}$$

3. Find the dimensions of the box that requires the least material for construction of the box being opened at the top and having a volume 32cc.

Solution :

Let x, y, z be the length, breadth and height of the box.

Then surface area of the box = $xy + 2yz + 2zx$, since the box is opened at the top.

Given, volume should be 32.

Therefore, $xyz = 32 \rightarrow xyz - 32 = 0$

Thus $F(x, y, z) = (xy + 2yz + 2zx) + \lambda (xyz - 32) \rightarrow (1)$

$$\frac{\partial F}{\partial x} = y + 2z + \lambda(yz)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda(zx)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda(xy)$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32$$

$$\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = \frac{-y - 2z}{yz} = \frac{-x - 2z}{xz} = \frac{-2y - 2x}{xy}$$

$$\left.\begin{aligned} &\rightarrow \frac{-y-2z}{yz} = \frac{-x-2z}{xz} \Rightarrow x = y \\ &\rightarrow \frac{-x-2z}{xz} = \frac{-2y-2x}{xy} \Rightarrow y = 2z \end{aligned}\right\} \quad \text{x = y = 2z.}$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz - 32 = 0$$

⇨ $x \times x \times \frac{x}{2} = 32$

⇨ x = 4

⇨ y = 4 and z = 2.

Thus the dimension of the box is (4, 4, 2)

4. Find the minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$.

Solution :

Let (x, y, z) be any point on the cone $x^2 + y^2 = 4z^2$.

Then its distance from the point (3, 4, 15) is

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-15)^2.}$$

First we find the minimum value of d^2 subject to the condition $x^2 + y^2 = 4z^2$.

Let $F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-15)^2 + \lambda\,(x^2 + y^2 - 4z^2)$

The stationary points are given by,

$F_x = 2(x - 3) + 2\lambda x = 0$ --------------> (1)

$F_y = 2(y - 4) + 2\lambda y = 0$ --------------> (2)

$F_z = 2(z - 15) - 8\lambda z = 0$ --------------> (3)

$F_\lambda = x^2 + y^2 - 4z^2 = 0$ --------------> (4)

From (1), $x = \frac{3}{1+\lambda}$

From (2), $y = \frac{4}{1+\lambda}$

From (3),$z = \frac{15}{1-4\lambda}$

Substituting in (4), $\left(\frac{3}{1+\lambda}\right)^2 + \left(\frac{4}{1+\lambda}\right)^2 = 4\left(\frac{15}{1-4\lambda}\right)^2$

i.e., $25(1 - 4\lambda)^2 = 4.225\ (1 + \lambda)^2$

i.e., $\frac{1-4\lambda}{1+\lambda} = \pm 6$

From $\frac{1-4\lambda}{1+\lambda} = 6$ we get $\lambda = {}^{-1}/_{2}$

From $\frac{1-4\lambda}{1+\lambda} = -6$ we get $\lambda = {}^{-7}/_{2}$

When $\lambda = {}^{-1}/_{2}$, we get x = 6, y = 8, z = 5.

When $\lambda = {}^{-7}/_{2}$, we get x = -6/5, y = -8/5, z = 1

Thus the stationary points are (6, 8, 5) and (-6/5, -8/5, 1).

Distance of (6, 8, 5) from (3, 4, 15) is $d = \sqrt{(6-3)^2 + (8-4)^2 + (5-15)^2}$

$$= \sqrt{125} \quad = 5\sqrt{5}$$

Distance of (-6/5, -8/5, 1) from (3, 4, 15) is

$$d = \sqrt{({}^{-6}/_{5} - 3)^2 + ({}^{-8}/_{5} - 4)^2 + (1 - 15)^2}$$

$$= \sqrt{\frac{441}{25} + \frac{784}{25} + 196} = \sqrt{49 + 196} = \sqrt{245} = 7\sqrt{5}$$

∴ The minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$ is $5\sqrt{5}$.

5. Find the volume of the greatest parallelepiped which has its sides parallel to co-ordinate planes and inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution:

Let 2x, 2y, 2z be the dimension of the rectangular parallelepiped.

We have to maximize 8xyz subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Therefore $F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$

$$\frac{\partial F}{\partial x}=8yz+\frac{2\lambda x}{a^2};$$

$$\frac{\partial F}{\partial y}=8xz+\frac{2\lambda y}{b^2}$$

$$\frac{\partial F}{\partial z}=8xy+\frac{2\lambda z}{c^2}$$

$$\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \lambda = \frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z}$$

Choosing $\frac{a^2yz}{x} = \frac{b^2xz}{y} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$

Choosing $\frac{b^2xz}{y} = \frac{c^2xy}{z} \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2}$

Thus $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$

$$\frac{\partial F}{\partial \lambda} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$3\frac{x^2}{a^2} = 1 \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, we can prove $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$

Thus the maximum volume is V = 8xyz $= \frac{8abc}{3\sqrt{3}}$.

6. Find the shortest and longest distance from (1,2,-1) to the sphere $x^2 + y^2 + z^2 = 24$, using lagrange's method of maxima and minima.

Solution:

Let (x,y,z) be any point on the sphere.

The distance from (1,2,-1) to the point (x,y,z) is given by

$$d = \sqrt{(x-1)^2 + (y-1)^2 + (z+1)^2}$$

$$d^2 = (x-1)^2 + (y-1)^2 + (z+1)^2$$

Now the problem is to optimize $d^2 = (x-1)^2 + (y-1)^2 + (z+1)^2$

Subject to $x^2 + y^2 + z^2 = 24$

Let $F(x, y, z) = (x-1)^2 + (y-1)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$

The stationary points are given by,

$F_x = 2(x - 1) + 2\lambda x = 0$ -------(1)
$F_y = 2(y - 2) + 2\lambda y = 0$ -------(2)
$F_z = 2(z +1) + 2\lambda z = 0$ --------(3)
$F_\lambda = x^2 + y^2 +z^2 -24=0$ ---------(4)

From (1), $x = \frac{1}{1+\lambda}$

From (2), $y = \frac{2}{1+\lambda}$

From (3),$z = \frac{1}{1+\lambda}$

Substituting in (4),$\frac{6}{(1+\lambda)^2} = 24$

(i.e.,) $(1 + \lambda)^2 = \frac{1}{4}$

$$Therefore\ \lambda = \frac{1}{2}\ or\ \frac{-3}{2}$$

When $\lambda = {}^{1}/_{2}$, the point on the sphere is (2,4,-2)

When $\lambda = {}^{-3}/_{2}$, the point on the sphere is (-2,-4,2)

At (2,4,-2) $d = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} \; = \; \sqrt{6}$

At (-2,-4,2) $d = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2} \; = \; 3\sqrt{6}$

Therefore shortest and the longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.